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Wave propagation in compressible hyperelastic solids in the presence of powerless internal forces

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Dedicated to the memory of Professor Kumbakonam R. Rajagopal.

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ABSTRACT

Although nonlinear elastodynamics has been widely studied since the second half of the last century, still nowadays, some aspects of this theory need a more systematic analysis. Based on the classical theory of simple materials of differential type and the results on the analytical form of constitutive models consistent with the laws of thermodynamics, we here analyze the wave propagation in compressible hyperelastic materials in the case in which the effects due to powerless internal forces are not negligible. We show that powerless internal forces affect the wave propagation in compressible hyperelastic solids by contributing to nonlinear wave dispersion and allowing the propagation of traveling waves.

1. Introduction

Nonlinear elastostatics is a mathematically elegant and rigorous continuum theory that has been used in a number of real world applications, providing predictions in good accordance with experimental evidences. On the contrary, its dynamic counterpart is a complex and discussed theory. This is mainly due to the fact that in modeling the dynamic response of real bodies dispersive and/or dissipative effects cannot be neglected. Obviously, including dispersion and/or dissipation in the modeling of the mechanical response of materials makes the mathematical structure of the equations governing the motion quite intricate.

Despite the complexity of the equations of nonlinear elastodynamics, several aspects of wave propagation in dispersive solids have been investigated since the second half of the past century. Reporting a complete list of the works that address this topic represents a hard task and goes beyond the scope of this paper. Here, we limit to quote two classical reference sources (Achenbach, 2012; Graff, 2012) in which several aspects of wave propagation in solids are thoroughly studied.

In the last two decades, by using the model for material dispersion proposed by Rubin, Rosenau, and Gottlieb (1995), Saccomandi and co-workers (Destrade & Saccomandi, 2006, 2008; Saccomandi & Vergori, 2023) showed that coherent structures such as periodic or solitary waves, or pulse waves with compact support can propagate in a dispersive hyperelastic solid. Recently, based on the results of Edelen (Edelen, 1973, 1974, 1977) on the form of the constitutive relations consistent with the second law of thermodynamics, Saccomandi and his research group (Amendola, de Castro Motta, Saccomandi, & Vergori, 2024; Saccomandi & Vergori, 2024) have introduced constitutive models for the Cauchy stress tensor that account for material dispersion in both isotropic and anisotropic materials. In particular, a very general model for the Cauchy stress tensor of isotropic dispersive hyperelastic materials has been introduced within the theory of simple material of second grade (Truesdell & Noll, 2004). Assuming that dissipative effects are negligible, the model proposed in Saccomandi and Vergori (2024) consists in the sum of a purely elastic and a dispersive parts, with

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the latter resulting, in turn, in the sum of two contributions: one deriving from a potential and the other accounting for powerless internal forces.

In Saccomandi and Vergori (2024) the theory developed for slightly dispersive quasi-incompressible hyperelastic materials has been used for determining kink and solitary waves propagating in shape memory alloys, but a deeper analysis of the contributions to the wave propagation due to the powerless internal forces has not been carried out. In this manuscript, we aim at providing a more detailed study of the powerless part of the Cauchy stress tensor under the assumption that the dispersive part of the stress tensor associated with a potential introduced in Saccomandi and Vergori (2024) can be neglected. Specifically, we shall prove that the contributions due to powerless internal forces manifests only at large enough strain rates and, in conjunction with nonlinearity, cause wave dispersion, and allow the propagation of traveling wave. For these reasons, it is appropriate to refer to the powerless contribution to the stress tensor as powerless dispersive stress tensor.

The plan of the paper is as follows. In Section 2 we recall the basic equations and notions in nonlinear elasticity, and introduce a very general model for the Cauchy stress tensor that accounts for the contributions due to powerless internal forces within the theory of simple materials of the second grade. In Sections 3 and 4 we determine the approximation of this model valid at small but finite strains, strain rates and rates of change of the strain rates. We then derive the governing equations in the one dimensional case (Section 5), and prove the existence of global (in time and space) solutions of the wave equations of the Carroll type (Section 6). In Sections 7 and 8 we study, respectively, the interaction between transverse and longitudinal waves and discuss the existence of traveling wave solutions within the fourth-order theory of elasticity. Specifically, considering the model for the strain energy function of carbon black-reinforced natural rubber proposed in Cantournet, Layouni, Laiarinandrasana, and Piques (2014), we prove the transverse and longitudinal periodic waves propagate with the same wavelength. We finally conclude with some remarks (Section 9).

2. Basic equations

Let $\mathcal{B}$ be a body, namely a set whose elements (called particles) are in one-to-one correspondence with a region $\mathscr{B}$ of the three-dimensional Euclidean point space $\mathcal{E}$. $\mathscr{B}$ is referred to as a configuration of $\mathcal{B}$. Since the body $\mathcal{B}$ occupies different configurations at different times, to study the evolution in time of the positions of the particles of $\mathcal{B}$ it is convenient to introduce a reference configuration $\mathscr{B}_r$ and identify the particles of $\mathcal{B}$ with the position vectors (relative to an origin $O \in \mathcal{E}$) of the points of $\mathscr{B}_r$. In other words, we identify a particle $P \in \mathcal{B}$ with the position vector $\boldsymbol{X} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k}$ of the point $(X, Y, Z) \in \mathscr{B}_r$ in one-to-one correspondence with P. The choice of the reference configuration is completely arbitrary. Here, for convenience of presentation, we choose $\mathscr{B}_r$ as the configuration occupied by $\mathcal{B}$ at the initial time $t = 0$. With this choice, we define a motion of $\mathcal{B}$ in the time interval $[0, T)$ as a mapping $\chi : \mathscr{B}_r \times [0, T) \to \mathcal{E}$ such that, for any $t \in (0, T)$, $\chi_t \equiv \chi(\cdot, t)$ is one-to-one and $\boldsymbol{x} = \chi(\boldsymbol{X}, t) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ is the position vector (relative to the same origin O) of the particle P at time $t > 0$. The configuration of the body at time t, $\mathscr{B}_t = \chi_t(\mathscr{B}_r) = \chi(\mathscr{B}_r, t)$, is the current configuration.

The vector function

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} = \chi(\boldsymbol{X}, t) - \boldsymbol{X} \tag{1}$$

is the displacement field, and the second-order tensors

$$\boldsymbol{H} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \quad \text{and} \quad \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \tag{2}$$

are the displacement and deformation gradients. From (1) and (2) it is easy to deduce that the displacement and deformation gradients are related through the identity $\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H}$. In terms of $\boldsymbol{F}$, the right and left Cauchy–Green deformation tensors $\boldsymbol{C}$ and $\boldsymbol{B}$, and the Green–Lagrange strain tensor read

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}, \quad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T \quad \text{and} \quad \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \widetilde{\boldsymbol{E}} + \frac{1}{2}\boldsymbol{H}^T\boldsymbol{H}, \tag{3}$$

where $\widetilde{\boldsymbol{E}} = (\boldsymbol{H} + \boldsymbol{H}^T)/2$ is the infinitesimal strain tensor.

The left and right Cauchy–Green deformation tensors satisfy the same characteristic equation as their principal scalar invariants coincide, namely

$$I_1 \equiv \operatorname{tr}\boldsymbol{C} = \operatorname{tr}\boldsymbol{B}, \tag{4a}$$

$$I_2 \equiv \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{C})^2 - \operatorname{tr}(\boldsymbol{C}^2)\right] = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{B})^2 - \operatorname{tr}(\boldsymbol{B}^2)\right], \tag{4b}$$

$$I_3 \equiv \det\boldsymbol{C} = \det\boldsymbol{B} = (\det\boldsymbol{F})^2 \equiv J^2, \tag{4c}$$

whence $\boldsymbol{C}$ and $\boldsymbol{B}$ have the same spectrum, with their eigenvalues being the square of the principal stretches.

The principal scalar invariants (4) are related to the invariants $\mathcal{J}_n = \operatorname{tr}(\boldsymbol{E}^n)$ $(n = 1, 2, 3)$ of the Green–Lagrange strain tensor through the identities

$$I_1 = 2\mathcal{J}_1 + 3, \quad I_2 = 2\mathcal{J}_1^2 - 2\mathcal{J}_2 + 4\mathcal{J}_1 + 3, \tag{5a}$$

and

$$I_3 = \frac{8}{3}\mathcal{J}_3 - 4\mathcal{J}_1\mathcal{J}_2 + \frac{4}{3}\mathcal{J}_1^3 + 2\mathcal{J}_1^2 - 2\mathcal{J}_2 + 2\mathcal{J}_1 + 1. \tag{5b}$$

Any scalar, vector, or tensor field $\boldsymbol{\Phi}$ associated with the kinematics of the body $\mathcal{B}$ can be expressed both as a function of particles $\boldsymbol{X}$ and time t and in terms of positions $\boldsymbol{x}$ and time t. In the former case, $\boldsymbol{\Phi}$ has domain $\mathscr{B}_r \times [0,T)$ and is called a material field; in the latter, $\boldsymbol{\Phi}$ is a spatial field with domain $\mathscr{B}_t \times [0,T)$. Given a material field $\boldsymbol{\Phi}$, the spatial description $\boldsymbol{\Phi}_s$ of $\boldsymbol{\Phi}$ is defined as

$$\boldsymbol{\Phi}_s : (\boldsymbol{x},t) \in \mathscr{B}_t \times [0,T) \mapsto \boldsymbol{\Phi}(\chi_t^{-1}(\boldsymbol{x}),t). \tag{6}$$

Analogously, the material description of Ξ_m of a spatial field Ξ is given by

$$\Xi_m : (\boldsymbol{X},t) \in \mathscr{B}_r \times [0,T) \mapsto \Xi(\chi_t(\boldsymbol{x}),t). \tag{7}$$

Denoted by $\rho_0 = \hat{\rho}_0(\boldsymbol{X})$ and $\rho = \hat{\rho}(\boldsymbol{x},t)$ the mass densities in the reference and current configurations, respectively, from the conservation of mass and the change of variables (7) it follows that

$$\rho_0 = \rho_m J. \tag{8}$$

In what follows, for simplicity, we assume that the reference mass density is constant, i.e., the body is homogeneous in the reference configuration.

Introduced the spatial description of the velocity field

$$\boldsymbol{v} : (\boldsymbol{x},t) \in \mathcal{B}_t \times [0,T] \mapsto \frac{\partial \chi}{\partial t}(\chi_t^{-1}(\boldsymbol{x}),t), \tag{9}$$

it can be proven that the velocity gradient $\boldsymbol{L} = \partial \boldsymbol{v}/\partial \boldsymbol{x}$ is related to the deformation gradient and its rate of change through the relation

$$\boldsymbol{L}_m = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \tag{10}$$

where, henceforth, the superposed dot and, when necessary to avoid confusion, the differential operator $\mathrm{D}/\mathrm{D}t$ denote the material time derivative.

Also the rate of change of the determinant of the deformation gradient J (and, obviously, the rate of change of the volume strain $J-1$) is related to the velocity gradient. Indeed, from (10) it can be proven that

$$\dot{J} = J \operatorname{tr}(\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}) = J \operatorname{tr} \boldsymbol{L}_m = J(\operatorname{div}\boldsymbol{v})_m. \tag{11}$$

The symmetric part of the velocity gradient $\boldsymbol{D} = (\boldsymbol{L} + \boldsymbol{L}^T)/2$ represents the strain-rate tensor, and the skew-symmetric part $\boldsymbol{W} = (\boldsymbol{L} - \boldsymbol{L}^T)/2$ is the spin tensor. As it is well known, the strain-rate tensor is objective, whereas the body spin is not an objective tensor field (Ogden, 1984).

Given $n \in \mathbb{N}$, we define the nth Rivlin–Ericksen $\boldsymbol{A}_n$ tensor as follows

$$\boldsymbol{A}_n = \boldsymbol{F}^{-T} \frac{\mathrm{D}^n \boldsymbol{C}}{\mathrm{D}t^n} \boldsymbol{F}^{-1}. \tag{12}$$

It is easy to check that

$$\boldsymbol{A}_1 = \boldsymbol{L}_m + \boldsymbol{L}_m^T = 2\boldsymbol{D}_m, \tag{13a}$$

and, for any $n \in \mathbb{N} - \{1\}$, the $(n+1)$th Rivlin–Ericksen tensor can be computed through the recursive formula

$$\boldsymbol{A}_{n+1} = \overset{\circ}{\boldsymbol{A}}_n + \boldsymbol{D}_m \boldsymbol{A}_n + \boldsymbol{A}_n \boldsymbol{D}_m, \tag{13b}$$

where the superposed circle denotes the co-rotational time derivative (Saccomandi & Vergori, 2024). From (13a) and (13b) one deduces that the Rivlin–Ericksen tensors (12) are objective.

In the next sections, we consider simple materials of differential type and, specifically, introduce constitutive relations for the stress tensor that depend only on the first two Rivlin–Ericksen tensors. From (13a) and (13b) it is easy to deduce that these two tensors are related through

$$\boldsymbol{A}_2 = \overset{\circ}{\boldsymbol{A}}_1 + \boldsymbol{A}_1^2, \tag{14}$$

which combined with (11) yields the identities

$$\operatorname{tr} \boldsymbol{A}_2 = \frac{\mathrm{D}}{\mathrm{D}t}(\ln J) + \operatorname{tr} \boldsymbol{A}_1^2 \tag{15a}$$

and

$$\boldsymbol{A}_2 \cdot \boldsymbol{A}_1 = \operatorname{tr}(\boldsymbol{A}_2 \boldsymbol{A}_1) = \frac{1}{2} \frac{\mathrm{D}}{\mathrm{D}t}(\operatorname{tr} \boldsymbol{A}_1^2) + \operatorname{tr} \boldsymbol{A}_1^3. \tag{15b}$$

Assuming valid Cauchy's postulate on the internal actions in a continuum, introducing the spatial description of the acceleration field

$$\boldsymbol{a} : (\boldsymbol{x},t) \in \mathcal{B}_t \times [0,T] \mapsto \frac{\partial^2 \chi}{\partial t^2}(\chi_t^{-1}(\boldsymbol{x}),t), \tag{16}$$

and denoting $\boldsymbol{T}$ the Cauchy stress tensor, in the absence of body forces the equations of balance of linear and angular momenta yield, respectively, the equation of motion

$$\rho \boldsymbol{a} = \mathrm{div}\boldsymbol{T} \tag{17}$$

and the symmetry of $\boldsymbol{T}$.

The equation of motion (17) is in Eulerian form as $\boldsymbol{a}$ and $\boldsymbol{T}$ are spatial fields and the divergence operator div contains partial derivatives with respect to the spatial coordinates x_1, x_2 and x_3. In what follows, we find it more convenient to write the equations of motion in Lagrangian form in which all the terms are material vector fields. Then, introduced the first Piola–Kirchhoff stress tensor

$$\boldsymbol{P} = J\boldsymbol{T}_m\boldsymbol{F}^{-T}, \tag{18}$$

and denoted Div the divergence operator expressed in terms of the partial derivatives with respect to the material coordinates X_1, X_2 and X_2, the equation of motion in Lagrangian form reads

$$\rho_0 \frac{\partial^2 \boldsymbol{\chi}}{\partial t^2} = \mathrm{Div}\boldsymbol{P}. \tag{19}$$

Hereinafter, to make the analysis as simple as possible, we shall limit our analysis to homogeneous bodies for which ρ_0 is constant.

2.1. A model for the Cauchy stress tensor

We now propose a model for the Cauchy stress tensor that accounts for the effects due to powerless internal actions in a compressible isotropic hyperelastic solid. For simplicity, we divide the Cauchy stress tensor into the sum of two independent contributions: a purely elastic part $\boldsymbol{T}_E = \hat{\boldsymbol{T}}_E(\boldsymbol{B})$ associated with an elastic potential energy, and a powerless part $\boldsymbol{T}_0 = \hat{\boldsymbol{T}}_0(\boldsymbol{A}_1, \boldsymbol{A}_2)$ orthogonal to the strain-rate tensor. In other words, based on the results of Edelen (Edelen, 1973, 1974, 1977), we assume that the Cauchy stress tensor is of the form

$$\boldsymbol{T} = \boldsymbol{T}_E + \boldsymbol{T}_0, \tag{20}$$

where $\boldsymbol{T}_E \cdot \boldsymbol{D} = \rho\dot{e}$, e is the specific elastic potential energy, and $\boldsymbol{T}_0 \cdot \boldsymbol{D} = 0$.

Within the theory of isotropic hyperelastic solids, the elastic part $\boldsymbol{T}_E$ is derived from the strain energy density per unit undeformed volume $\mathcal{W} = \mathcal{W}(I_1, I_2, I_3)$ (related to the specific elastic potential through the identity $e = \mathcal{W}/\rho_0$ Saccomandi & Vergori, 2024) by the relation

$$\boldsymbol{T}_E = 2I_3^{-1/2}\left(I_2\mathcal{W}_2 + I_3\mathcal{W}_3\right)\boldsymbol{I} + 2I_3^{-1/2}\mathcal{W}_1\boldsymbol{B} - 2I_3^{1/2}\mathcal{W}_2\boldsymbol{B}^{-1}, \tag{21}$$

where, henceforth, the subscript $i = 1, 2, 3$ appended to the strain energy density denotes differentiation of $\mathcal{W}$ with respect to the invariant I_i (Ogden, 1984).

Since energy can be defined up to an additive constant, for convenience, we take the strain energy density as zero in the reference configuration. In addition, excluding situations in which residual stresses are non-negligible, we assume that the reference configuration is stress-free. These two facts lead to the restrictions

$$\overline{\mathcal{W}} = 0 \quad \text{and} \quad \overline{\mathcal{W}}_1 + 2\overline{\mathcal{W}}_2 + \overline{\mathcal{W}}_3 = 0, \tag{22}$$

where, henceforth, the bars placed over the strain energy function $\mathcal{W}$ and its partial derivatives indicate that the quantities underneath are evaluated at the reference configuration (for which $I_1 = I_2 = 3$ and $I_3 = 1$).

As the powerless part of the Cauchy stress is concerned, within the theory of simple materials of second grade, Saccommandi and Vergori (Saccomandi & Vergori, 2024) proved that material description of $\boldsymbol{T}_0$ is of the form

$$\boldsymbol{T}_{0m} = \rho_m\psi(\delta_1, \delta_2, \delta_3)\left(2\delta_3\boldsymbol{I} - \delta_1\boldsymbol{A}_1\right) \tag{23}$$

where the response function ψ has physical dimensions $[L^2]$, and

$$\delta_1 = \frac{\mathrm{tr}\boldsymbol{A}_1}{2}, \quad \delta_2 = \frac{\mathrm{tr}\boldsymbol{A}_2}{2} \quad \text{and} \quad \delta_3 = \frac{\mathrm{tr}\boldsymbol{A}_1^2}{4} \tag{24}$$

are invariants of the first two Rivlin–Ericksen tensors.

For incompressible materials, the powerless part of the Cauchy stress (23) does not occur because of the incompressibility constraint $\delta_1 = 0$ and the necessity to introduce a reactive part of the Cauchy stress of the form $\boldsymbol{T}_C - p\boldsymbol{I}$, with the arbitrary scalar field p being a Lagrange multiplier. In simple terms, for incompressible materials, the powerless part $\boldsymbol{T}_0$ is absorbed by the arbitrary reactive part $\boldsymbol{T}_C$.

3. Linear, weakly nonlinear and fully nonlinear models for the strain energy function

Under the assumption of smallness of the components of the displacement gradient $\boldsymbol{H}$, that is

$$\epsilon = \max_{i,j=1,2,3} \max_{(\boldsymbol{X},t)\in\mathcal{B}_r\times[0,T)} |H_{ij}(\boldsymbol{X},t)| \ll 1, \tag{25}$$

it can be proven that, to the leading order, the Maclaurin expansion of the strain energy function $\mathcal{W}$ expressed in terms of the invariants $\mathcal{J}$'s of the Green–Lagrange strain tensor through the connections (5) gives the strain energy function in the linearized theory of elasticity

$$\mathcal{W} \approx \frac{\lambda}{2}\mathcal{J}_1^2 + \mu\mathcal{J}_2 + O(\epsilon^3), \tag{26}$$

with the Lamé moduli λ and μ being related to the derivatives of $\mathcal{W}$ evaluated at the reference configuration according to equations (101a) and (101b) in Saccomandi and Vergori (2021). One can prove that, for the undeformed configuration to be a local minimum of (26), it is necessary and sufficient that

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0. \tag{27}$$

For small but finite deformations, the Maclaurin expansion of $\mathcal{W}$ up to and including terms of order $O(\epsilon^4)$ yields the strain energy function within the fourth-order theory of elasticity

$$\mathcal{W} \approx \frac{\lambda}{2}\mathcal{J}_1^2 + \mu\mathcal{J}_2 + \frac{A}{3}\mathcal{J}_3 + B\mathcal{J}_1\mathcal{J}_2 + \frac{C}{3}\mathcal{J}_1^3 + E\mathcal{J}_1\mathcal{J}_3 + F\mathcal{J}_1^2\mathcal{J}_2 + G\mathcal{J}_2^2 + H\mathcal{J}_1^4 + O(\epsilon^5), \tag{28}$$

where the third- and fourth-order Landau constants A, B, C, E, F, G and H are as in equations (101c)-(101i) in Saccomandi and Vergori (2021).

For large deformations, the choice of the functional form of the strain energy that guarantees reasonable material behavior (the *hauptproblem* in nonlinear elasticity as Truesdell (Truesdell, 1956) defined it) opened a debate that is still open. We refer the reader to Saccomandi, Vergori, and Zanetti (2022) for a detailed discussion of this point. This explains why there are a number of mathematical models for the strain energy function of isotropic elastic solids. The only way to choose a suitable functional form of $\mathcal{W}$ consists in testing the theoretical predictions against experimental data. Since our study is purely theoretical, in the subsequent sections, just for illustration of our results, we will make use of the following model proposed by Cantournet et al. (2014) for carbon black-reinforced natural rubber:

$$\begin{aligned}\mathcal{W} &= C_{10}(I_3^{-1/3}I_1 - 3) + C_{01}(I_3^{-2/3}I_2 - 3) + C_{20}(I_3^{-1/3}I_1 - 3)^2 \\ &\quad + \frac{\kappa}{4}\left(I_3 - \ln I_3 - 1\right),\end{aligned} \tag{29a}$$

where

$$C_{10} = 0.15\,\text{MPa}, \quad C_{01} = 0.2\,\text{MPa}, \quad C_{20} = 0.0235\,\text{MPa}, \quad \kappa = 750\,\text{MPa}. \tag{29b}$$

4. Powerless dispersion tensor within the fourth-order theory of elasticity

In this section we derive the approximation for the powerless dispersion tensor that is valid at small strains, strain rates and rates of change of the strain rate.

To this aim, introduced the maximum modulus of the displacement field and the speed of propagation of transverse waves within the linearized theory of elasticity

$$l = \max_{(\boldsymbol{X},t)\in\mathcal{B}_r\times[0,T]} |\boldsymbol{u}(\boldsymbol{X},t)| \quad \text{and} \quad c_T = \sqrt{\frac{\mu}{\rho_0}}, \tag{30}$$

respectively, we assume that

$$O\left(\frac{l}{c_T}\max_{i,j=1,2,3}\max_{(\boldsymbol{X},t)\in\mathcal{B}_r\times[0,T)} |\dot{H}_{ij}(\boldsymbol{X},t)|\right) = O(\epsilon) \tag{31a}$$

and

$$O\left(\frac{l^2}{c_T^2}\max_{i,j=1,2,3}\max_{(\boldsymbol{X},t)\in\mathcal{B}_r\times[0,T)} |\ddot{H}_{ij}(\boldsymbol{X},t)|\right) = O(\epsilon), \tag{31b}$$

with ϵ as in (25).

Next, from (10), (13a) and (14) we deduce that to the leading order, the first two Rivlin–Ericksen tensors read

$$\boldsymbol{A}_1 = 2\tilde{\tilde{\boldsymbol{E}}} + \frac{c_T}{l}O\left(\epsilon^2\right) \quad \text{and} \quad \boldsymbol{A}_2 = 2\tilde{\tilde{\boldsymbol{E}}} + \frac{c_T^2}{l^2}O\left(\epsilon^2\right), \tag{32}$$

whence

$$\frac{l\delta_1}{c_T} = \underbrace{\frac{l}{c_T}\text{tr}\tilde{\tilde{\boldsymbol{E}}}}_{=O(\epsilon)} + O\left(\epsilon^2\right), \tag{33a}$$

$$\frac{l^2\delta_2}{c_T^2} = \underbrace{\frac{l^2}{c_T^2}\mathrm{tr}\tilde{\tilde{\boldsymbol{E}}}}_{=O(\epsilon)} + O\left(\epsilon^2\right) \tag{33b}$$

and

$$\frac{l^2\delta_3}{c_T^2} = \underbrace{\frac{l^2}{c_T^2}(\mathrm{tr}\tilde{\tilde{\boldsymbol{E}}})^2}_{=O(\epsilon^2)} + O\left(\epsilon^3\right). \tag{33c}$$

Finally, from (8), (32) and (33) we can conclude that up to and including terms of the third order in ϵ, the powerless dispersion tensor (23) approximates as

$$\boldsymbol{T}_{0m} = \rho_0\left[\psi_0(1-\mathcal{J}_1) + \psi_1\delta_1 + \psi_2\delta_2\right](2\delta_3\boldsymbol{I} - \delta_1\boldsymbol{A}_1) \tag{34}$$

where

$$\psi_0 = \psi(0,0,0) \quad \text{and} \quad \psi_k = \frac{\partial\psi}{\partial\delta_k}(0,0,0) \quad (k=1,2). \tag{35}$$

The approximation (34) can be regarded as the powerless dispersion tensor within the fourth-order theory of elasticity, with (35) representing dispersion moduli.

5. Wave equations

In what follows, we shall study the wave propagation in a compressible isotropic hyperelastic material in the presence of powerless internal actions. In particular, we aim at analyzing the interaction between longitudinal and transverse waves. To this aim, for simplicity, we consider the following class of motions

$$x = X + u(Z,t), \qquad y = Y + v(Z,t), \qquad z = Z + w(Z,t), \tag{36}$$

where $u(Z,t)$ and $v(Z,t)$ are transverse displacements, and $w(Z,t)$ represents the longitudinal displacement.

Using the subscript notation for differentiation, for this class of motions the invariants (4) and (24) read

$$I_1 = u_Z^2 + v_Z^2 + \left(w_Z+1\right)^2 + 2, \tag{37a}$$

$$I_2 = u_Z^2 + v_Z^2 + 2\left(w_Z+1\right)^2 + 1, \tag{37b}$$

$$I_3 = (1+w_Z)^2 = J^2, \tag{37c}$$

$$\delta_1 = J^{-1}w_{Zt}, \tag{37d}$$

$$\delta_2 = J^{-2}(u_{Zt}^2 + v_{Zt}^2 + w_{Zt}^2) + J^{-1}w_{Ztt}, \tag{37e}$$

$$\delta_3 = J^{-2}[(u_{Zt}^2 + v_{Zt}^2)/2 + w_{Zt}^2], \tag{37f}$$

and from (20), (21) and (23) the Piola–Kirchhoff stress tensor (18) has components

$$\begin{aligned} P_{11} &= Q_1(u_Z^2 + v_Z^2, w_Z) + J^2Q_2(u_Z^2 + v_Z^2, w_Z) + 2\mathcal{W}_2(I_1,I_2,I_3)v_Z^2 \\ &\quad + \rho_0\psi(\delta_1,\delta_2,\delta_3)\left[J^{-2}\left(u_{Zt}^2 + v_{Zt}^2 + 2w_{Zt}^2\right) + J^{-3}u_Zu_{Zt}w_{Zt}\right], \end{aligned} \tag{38a}$$

$$P_{12} = v_Z[\rho_0J^{-3}\psi(\delta_1,\delta_2,\delta_3)u_{Zt}w_{Zt} - 2\mathcal{W}_2(I_1,I_2,I_3)u_Z], \tag{38b}$$

$$P_{13} = Q_1(u_Z^2 + v_Z^2, w_Z)u_Z - \rho_0J^{-3}\psi(\delta_1,\delta_2,\delta_3)u_{Zt}w_{Zt}, \tag{38c}$$

$$P_{21} = u_Z[\rho_0J^{-3}\psi(\delta_1,\delta_2,\delta_3)u_{Zt}w_{Zt} - 2\mathcal{W}_2(I_1,I_2,I_3)v_Z], \tag{38d}$$

$$\begin{aligned} P_{22} &= Q_1(u_Z^2 + v_Z^2, w_Z) + J^2Q_2(u_Z^2 + v_Z^2, w_Z) + 2\mathcal{W}_2(I_1,I_2,I_3)u_Z^2 \\ &\quad + \rho_0\psi(\delta_1,\delta_2,\delta_3)\left[J^{-2}\left(u_{Zt}^2 + v_{Zt}^2 + 2w_{Zt}^2\right) + J^{-3}v_Zv_{Zt}w_{Zt}\right], \end{aligned} \tag{38e}$$

$$P_{23} = Q_1(u_Z^2 + v_Z^2, w_Z)v_Z - \rho_0J^{-3}\psi(\delta_1,\delta_2,\delta_3)v_{Zt}w_{Zt}, \tag{38f}$$

$$P_{31} = -JQ_2(u_Z^2 + v_Z^2, w_Z)u_Z - \rho_0 J^{-2}\psi(\delta_1, \delta_2, \delta_3)\left[u_{Zt}w_{Zt} + J^{-1}u_Z\left(u_{Zt}^2 + v_{Zt}^2\right)\right], \tag{38g}$$

$$P_{32} = -JQ_2(u_Z^2 + v_Z^2, w_Z)v_Z - \rho_0 J^{-2}\psi(\delta_1, \delta_2, \delta_3)\left[v_{Zt}w_{Zt} + J^{-1}v_Z\left(u_{Zt}^2 + v_{Zt}^2\right)\right], \tag{38h}$$

$$P_{33} = J[Q_1(u_Z^2 + v_Z^2, w_Z) + Q_2(u_Z^2 + v_Z^2, w_Z)] + \rho_0 J^{-3}\psi(\delta_1, \delta_2, \delta_3)\left(u_{Zt}^2 + v_{Zt}^2\right), \tag{38i}$$

$$Q_1(u_Z^2 + v_Z^2, w_Z) = 2[\mathcal{W}_1(I_1, I_2, I_3) + \mathcal{W}_2(I_1, I_2, I_3)], \tag{38j}$$

where and

$$Q_2(u_Z^2 + v_Z^2, w_Z) = 2[\mathcal{W}_2(I_1, I_2, I_3) + \mathcal{W}_3(I_1, I_2, I_3)]. \tag{38k}$$

Then, specializing the equation of motion (19) to the class of motions (36), we obtain the following system of partial differential equations in the components of the displacement vector fields

$$\begin{cases} \rho_0 u_{tt} = \dfrac{\partial P_{13}}{\partial Z} \\ \rho_0 v_{tt} = \dfrac{\partial P_{23}}{\partial Z} \\ \rho_0 w_{tt} = \dfrac{\partial P_{33}}{\partial Z} \end{cases} \tag{39}$$

To ease the analysis of the system of governing equations (39), we find it more convenient to study the class of motions (36) in terms of the complex transverse strain $\Psi = U + iV \equiv u_Z + iv_Z$ and the longitudinal strain $W = w_Z = J - 1$. In so doing, in view of (37) the potential elastic energy per unit mass $\widetilde{\mathcal{W}}$ reads

$$\widetilde{\mathcal{W}}(|\Psi|^2, J) = \rho_0^{-1}\mathcal{W}(|\Psi|^2 + J^2 + 2, |\Psi|^2 + 2J^2 + 1, J^2), \tag{40}$$

and the equations of motion (39) become

$$\begin{cases} \Psi_{tt} = \dfrac{\partial^2}{\partial Z^2}\left[2\widetilde{Q}_1(|\Psi|^2, J)\Psi - J^{-3}\psi(\delta_1, \delta_2, \delta_3)\Psi_t W_t\right], \\ W_{tt} = \dfrac{\partial^2}{\partial Z^2}\left[\widetilde{Q}_2(|\Psi|^2, J) + J^{-3}\psi(\delta_1, \delta_2, \delta_3)|\Psi_t|^2\right], \end{cases} \tag{41}$$

where

$$\widetilde{Q}_1(|\Psi|^2, J) = \frac{\partial\widetilde{\mathcal{W}}}{\partial|\Psi|^2}(|\Psi|^2, J), \quad \widetilde{Q}_2(|\Psi|^2, J) = \frac{\partial\widetilde{\mathcal{W}}}{\partial J}(|\Psi|^2, J), \tag{42}$$

and the invariants (37d)–(37f) can be rewritten as

$$\delta_1 = J^{-1}W_t, \quad \delta_2 = J^{-2}(|\Psi_t|^2 + W_t^2) + J^{-1}W_{tt} \quad \text{and} \quad \delta_3 = J^{-2}\left(\frac{|\Psi_t|^2}{2} + W_t^2\right). \tag{43}$$

5.1. Approximate equations within the fourth-order theory of elasticity

Within the fourth-order weakly nonlinear theory of elasticity, from (22) and equations (101) in Saccomandi and Vergori (2021) the function $\widetilde{\mathcal{W}}$ approximates as

$$\widetilde{\mathcal{W}}(|\Psi|^2, J) = \frac{c_T^2}{2}|\Psi|^2 + \frac{c_L^2}{2}W^2 + \frac{\alpha_{LT}}{2}|\Psi|^2 W + \frac{\xi_L}{3}W^3 + \frac{\zeta_T}{4}|\Psi|^4 + \frac{\beta_{LT}}{2}|\Psi|^2 W^2 + \frac{\zeta_L}{4}W^4, \tag{44a}$$

where

$$c_T^2 = \frac{\mu}{\rho_0} = 2\frac{\partial\widetilde{\mathcal{W}}}{\partial(|\Psi|^2)}(0, 1) \tag{44b}$$

and

$$c_L^2 = \frac{\partial^2\widetilde{\mathcal{W}}}{\partial J^2}(0, 1) = \frac{\lambda + 2\mu}{\rho_0} \tag{44c}$$

are, respectively, the squares of the speeds of transverse and longitudinal waves within the linearized theory of elasticity, and, in view of (27), satisfy the inequality

$$c_L^2 \geq \frac{4}{3} c_T^2. \tag{44d}$$

The elastic moduli α_{LT}, ξ_L, ζ_T, β_{LT} and ζ_L are instead rescaled combinations of the third- and fourth-order Landau constants as

$$\alpha_{LT} = 2\frac{\partial^2 \widetilde{\mathcal{W}}}{\partial(|\Psi|^2)\partial J}(0,1) = \rho_0^{-1}\left(\lambda + 2\mu + \frac{A}{2} + B\right), \tag{44e}$$

$$\xi_L = \frac{1}{2}\frac{\partial^3 \widetilde{\mathcal{W}}}{\partial J^3}(0,1) = \rho_0^{-1}\left(\frac{3}{2}\lambda + 3\mu + A + 3B + C\right), \tag{44f}$$

$$\zeta_T = 2\frac{\partial^2 \mathcal{W}}{\partial(|\Psi|^2)^2}(0,1) = \rho_0^{-1}\left(\frac{\lambda}{2} + \mu + \frac{A}{2} + B + G\right), \tag{44g}$$

$$\begin{aligned}\beta_{LT} &= \frac{\partial^3 \mathcal{W}}{\partial(|\Psi|^2)\partial J^2}(0,1) \\ &= \rho_0^{-1}\left(\frac{\lambda}{2} + \mu + \frac{5}{4}A + \frac{7}{2}B + C + \frac{3}{2}E + F + 2G\right)\end{aligned} \tag{44h}$$

and

$$\begin{aligned}\zeta_L &= \frac{1}{6}\frac{\partial^4 \mathcal{W}}{\partial J^4}(0,1) \\ &= \rho_0^{-1}\left(\frac{\lambda}{2} + \mu + 2A + 6B + 2C + 4E + 4F + 4G + 4H\right).\end{aligned} \tag{44i}$$

Based on the results in Sections 3 and 4, within the fourth-order theory of elasticity the governing equations (39) approximate as

$$\begin{cases} \Psi_{tt} = \dfrac{\partial^2}{\partial Z^2}\Big[\, {c_T}^2\Psi + \alpha_{LT}\Psi W + \beta_{LT}\Psi W^2 + \zeta_T|\Psi|^2\Psi \\ \qquad\qquad -\psi_0(1-3\,W)\Psi_t W_t - \psi_1\Psi_t W_t^2 - \psi_2\Psi_t W_t W_{tt}\,\Big], \\ W_{tt} = \dfrac{\partial^2}{\partial Z^2}\Big[\, {c_L}^2\,W + \dfrac{\alpha_{LT}}{2}|\Psi|^2 + \beta_{LT}|\Psi|^2\,W + \xi_L W^2 + \zeta_L W^3 \\ \qquad\qquad +\psi_0(1-3\,W)|\Psi_t|^2 + \psi_1|\Psi_t|^2 W_t + \psi_2|\Psi_t|^2 W_{tt}\,\Big]. \end{cases} \tag{45}$$

6. A class of global wave solutions

Let us consider transverse waves with constant amplitudes propagating in a uniformly contracted or expanded medium. We then look for solutions in the form

$$\Psi = \mathcal{A}\exp[i\theta(Z,t)], \quad J_\star = 1 + W_\star = \text{constant}. \tag{46}$$

Inserting this ansatz in (45), dividing by the complex exponential in the equation for the (complex) transverse strain and taking its real and imaginary parts yield the following overdetermined system for the phase θ

$$\begin{cases} \theta_t^2 = 2\widetilde{Q}_1(\mathcal{A}^2, J_\star)\theta_Z^2, \\ \theta_{tt} = 2\widetilde{Q}_1(\mathcal{A}^2, J_\star)\theta_{ZZ}, \\ \dfrac{\partial^2}{\partial Z^2}\left[\psi\left(0, \mathcal{A}^2 J_\star^{-2}\theta_t^2, \mathcal{A}^2 J_\star^{-2}\theta_t^2/2\right)\right] = 0. \end{cases} \tag{47}$$

It is easy to show that this system admits a solution if and only if $\theta(Z,t) = kZ - \omega t$, with k and ω being constants such that

$$\frac{\omega^2}{k^2} = 2\widetilde{Q}_1(\mathcal{A}^2, J_\star). \tag{48}$$

Thus, if the amplitude $\mathcal{A}$ and the volume strain $W_\star$ are such that $\widetilde{Q}_1(\mathcal{A}, J_\star)$ is positive, and ω and k are positive constants whose ratio satisfies (48), then

$$\Psi_\pm = \mathcal{A}[\cos(kZ - \omega t) \pm i\sin(kZ - \omega t)], \quad W = W_\star, \tag{49}$$

are solutions of the governing equations (45). These solutions represent right-handed (Ψ_+) or left-handed ($\Psi\_$) circularly polarized transverse waves of Carroll type that propagate with phase speed $\omega/k = \sqrt{2\widetilde{Q}_1(\mathcal{A}, J_\star)}$ in a compressed (if $J_\star \in]0,1[$) or dilated (if $J_\star > 1$) solid.

For solids made of carbon black-reinforced natural rubber whose strain energy function is in the form (29) it is easy to check that $\widetilde{Q}_1(\mathcal{A}, J_\star) > 0$ for all $\mathcal{A} > 0$ and $J_\star > 0$, and, for any fixed value of the ratio of the deformed volume to the undeformed volume,

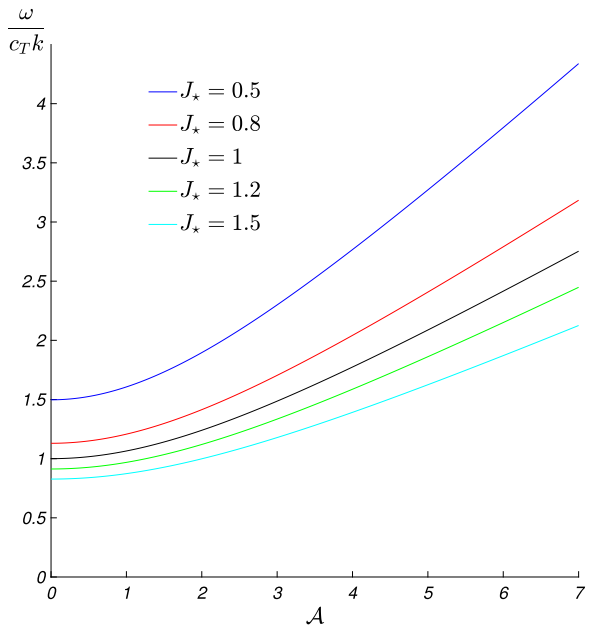


Fig. 1. Dimensionless phase speeds of Carroll-type waves in carbon black-reinforced natural rubber.

the phase speed of the circularly polarized waves of the Carroll-type increases as the amplitude of the wave itself increases. As it is reasonable to expect, the phase speed is higher in solids subjected to contraction than to expansion (see Fig. 1).

From (48) we readily deduce that the Carroll-type waves (49) are not affected by the powerless internal actions as they propagate with the same speed as in the absence of a powerless contribution to the Cauchy stress tensor. However, as well as being in closed form, these wave solutions are interesting from a purely mathematical point of view because they are global (both in time and in space) solutions of the Cauchy problem for (45) under the initial conditions

$$\begin{cases} \Psi(Z,0) = \mathcal{A}\left[\cos(kZ) \pm i\sin(kZ)\right], \\ \Psi_t(Z,0) = \mathcal{A}\omega\left[\sin(kZ) \mp i\cos(kZ)\right], \\ W(Z,0) = W_\star, \quad W_t(Z,0) = 0, \end{cases} \tag{50}$$

with $\mathcal{A}$, $W_\star$, k and ω being positive constants such that Eq. (48) is satisfied.

7. Interaction between transverse and longitudinal waves

We now study the interaction between transverse and longitudinal waves. To make the analytical treatment feasible, we limit our investigation to waves stimulated by an initial purely transverse excitation of small but finite amplitudes, and, in the framework of the fourth order theory of elasticity, find an approximation of the solution by using the method of multiple scales.

Since for initial purely longitudinal excitations of small amplitudes the structure of the system of wave Eqs. (41) allows solely the propagation of longitudinal waves which are not affected by the powerless contribution to the stress tensor, we here limit our analysis to waves stimulated by an initial purely transverse excitation.

7.1. Purely transverse initial excitation

To study the waves stimulated by a purely initial transverse excitation, we look for an approximate solution of the system of equations (41), with $\Phi = U + iV$, under the initial conditions

$$\begin{cases} U(Z,0) = \varepsilon U_0 \sin(kZ), \quad V(Z,0) = W(Z,0) = 0, \\ U_t(Z,0) = V_t(Z,0) = W_t(Z,0) = 0, \end{cases} \tag{51}$$

where k and ε are positive constants, with $0 < \varepsilon \ll 1$.

Since a standard perturbation analysis of Eqs. (41) would yield approximations that are valid only at not very large times because of the occurrence of secular terms, i.e. terms whose magnitude grows polynomially in time, we here prefer to use the method of multiple scales. In the absence of the powerless contribution to the stress tensor, the standard perturbation analysis of the wave Eqs. (41), with $\psi = 0$, produces approximations that are valid up to times of order $O(\varepsilon^{-1})$ because of the presence of terms that grows linearly in time in the transverse strain Saccomandi and Vergori (2023).

By following similar arguments as in Nayfeh and Mook (1995), we start by introducing the new independent variables

$$t_m = \varepsilon^m t, \tag{52}$$

with $m \in \mathbb{N}_0$, by which it follows that the derivatives with respect to time become expansions in the small parameter ε as

$$\frac{\partial}{\partial t} = \sum_{m=0}^{+\infty} \varepsilon^m \frac{\partial}{\partial t_m} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} = \sum_{m=0}^{+\infty} \varepsilon^m \left(\sum_{k=0}^{m} \frac{\partial^2}{\partial t_k \partial t_{m-k}} \right). \tag{53}$$

In addition, we assume that the transverse and longitudinal strains can be represented by expansions of the form

$$\Psi(Z,t;\varepsilon) = \sum_{n=1}^{+\infty} \varepsilon^n [U_n(Z,t_0,t_1,t_2,\dots) + iV_n(Z,t_0,t_1,t_2,\dots)], \tag{54a}$$

$$W(Z,t;\varepsilon) = \sum_{n=1}^{+\infty} \varepsilon^n W_n(Z,t_0,t_1,t_2,\dots). \tag{54b}$$

Here, we are interested in determining approximate solutions within the fourth order theory of elasticity. This means that only the dependence of the unknowns U_n, V_n and W_n in (54) on the independent variables t_0, t_1 and t_2 is needed. Substituting (53) and (54) into (41) and (51) and collecting terms of the same order in ε yield a collection of initial value problems (IVPs) from which we find the leading order approximation of the transverse and longitudinal waves stimulated by the initial excitation (51).

To order $O(\varepsilon)$ we have the following IVP

$$\frac{\partial^2 U_1}{\partial t_0^2} - c_T^2 \frac{\partial^2 U_1}{\partial Z^2} = 0, \tag{55a}$$

$$\frac{\partial^2 V_1}{\partial t_0^2} - c_T^2 \frac{\partial^2 V_1}{\partial Z^2} = 0, \tag{55b}$$

$$\frac{\partial^2 W_1}{\partial t_0^2} - c_L^2 \frac{\partial^2 W_1}{\partial Z^2} = 0, \tag{55c}$$

with

$$\begin{cases} U_1(Z,0,0,0) = U_0 \sin kZ, \quad V_1(Z,0,0,0) = W_1(Z,0,0,0) = 0, \\ \dfrac{\partial U_1}{\partial t_0}(Z,0,0,0) = \dfrac{\partial V_1}{\partial t_0}(Z,0,0,0) = \dfrac{\partial W_1}{\partial t_0}(Z,0,0,0) = 0. \end{cases} \tag{55d}$$

It is easy to check that the IVP (55) is solved by

$$U_1 = (F_1 \cos \omega_T t_0 + F_2 \sin \omega_T t_0) \sin kZ, \quad V_1 = W_1 = 0, \tag{56}$$

where $\omega_T = kc_T$ and $F_i = \hat{F}_i(t_1,t_2)$ $(i = 1,2)$ are arbitrary functions such that

$$\hat{F}_1(0,0) = U_0, \quad \hat{F}_2(0,0) = 0. \tag{57}$$

By taking into account (56), to order $O(\varepsilon^2)$ we obtain the IVP

$$\frac{\partial^2 U_2}{\partial t_0^2} - c_T^2 \frac{\partial^2 U_2}{\partial Z^2} = 2\omega_T \left(\frac{\partial F_1}{\partial t_1} \sin \omega_T t_0 - \frac{\partial F_2}{\partial t_1} \cos \omega_T t_0 \right) \sin kZ, \tag{58a}$$

$$\frac{\partial^2 V_2}{\partial t_0^2} - c_T^2 \frac{\partial^2 V_2}{\partial Z^2} = 0, \tag{58b}$$

$$\begin{aligned} \frac{\partial^2 W_2}{\partial t_0^2} - c_L^2 \frac{\partial^2 W_2}{\partial Z^2} = \frac{k^2}{2} \Big\{ & (2\omega_T^2 \psi_0 + \alpha_{LT})(F_1^2 + F_2^2) \\ & - (2\omega_T^2 \psi_0 - \alpha_{LT}) \Big[(F_1^2 - F_2^2) \cos 2\omega_T t_0 + 2F_1 F_2 \sin 2\omega_T t_0 \Big] \Big\} \cos 2kZ, \end{aligned} \tag{58c}$$

with

$$\begin{cases} U_2(Z,0,0,0) = V_2(Z,0,0,0) = W_2(Z,0,0,0) = 0, \\ \dfrac{\partial U_2}{\partial t_0}(Z,0,0,0) = -\dfrac{\partial U_1}{\partial t_1}(Z,0,0,0) = -\dfrac{\partial F_2}{\partial t_1}(0,0) \sin kZ, \\ \dfrac{\partial V_2}{\partial t_0}(Z,0,0,0) = \dfrac{\partial W_2}{\partial t_0}(Z,0,0,0) = 0. \end{cases} \tag{58d}$$

Observe that the solution for U_2 has secular terms containing the factors

$$t_0 \cos \omega_T t_0 \sin kZ \quad \text{and} \quad t_0 \sin \omega_T t_0 \sin kZ \tag{59}$$

unless

$$\frac{\partial F_i}{\partial t_1} = 0 \ (i = 1, 2). \tag{60}$$

Therefore, F_1 and F_2 must be independent of t_1, that is, $F_i = \hat{F}(t_2)$ $(i = 1, 2)$ and such that

$$\hat{F}_1(0) = U_0 \quad \hat{F}_2(0) = 0. \tag{61}$$

With (60) the solution of (58) is

$$U_2 = V_2 = 0, \tag{62a}$$

$$W_2 = \Bigg\{ F_3 \cos 2\omega_L t_0 + F_4 \sin 2\omega_L t_0 + \frac{k^2(2\psi_0\omega_T^2 + \alpha_{LT})}{8\omega_L^2}(F_1^2 + F_2^2) - \frac{k^2(2\psi_0\omega_T^2 - \alpha_{LT})}{8(\omega_L^2 - \omega_T^2)}\left[(F_1^2 - F_2^2)\cos 2\omega_T t_0 + 2F_1F_2 \sin 2\omega_T t_0\right] \Bigg\} \cos 2kZ \tag{62b}$$

where $\omega_L = kc_L$ and $F_k = \hat{F}_k(t_1, t_2)$ $(k = 3, 4)$ are arbitrary functions such that

$$\hat{F}_3(0,0) = \frac{k^2U_0^2[2\psi_0\omega_T^4 + \alpha_{LT}(\omega_T^2 - 2\omega_L^2)]}{8\omega_L^2(\omega_L^2 - \omega_T^2)} \equiv \Gamma U_0^2, \quad \hat{F}_4(0,0) = 0. \tag{63}$$

Observe that in view of (44d), the angular frequencies ω_T and ω_L satisfy the inequality

$$\omega_L^2 \geq \frac{4}{3}\omega_T^2 \tag{64}$$

which prevents the occurrence of resonant phenomena.

With (62), to order $O(\epsilon^3)$ we have that

$$\begin{aligned}
&\frac{\partial^2 U_3}{\partial t_0^2} - c_T^2 \frac{\partial^2 U_3}{\partial Z^2} = \Bigg[2\omega_T \left(\frac{\partial F_1}{\partial t_2} \sin \omega_T t_0 - \frac{\partial F_2}{\partial t_2} \cos \omega_T t_0 \right) \\
&+ a_1(F_1^2 + F_2^2)(F_1 \cos \omega_T t_0 + F_2 \sin \omega_T t_0) \\
&- a_4(F_1^2 - 3F_2^2)F_1 \cos 3\omega_T t_0 - a_4(3F_1^2 - F_2^2)F_2 \sin 3\omega_T t_0 \Bigg] \sin kZ \\
&+ \Big[9a_3(F_1^2 - 3F_2^2)F_1 \cos 3\omega_T t_0 + 9a_3(3F_1^2 - F_2^2)F_2 \sin 3\omega_T t_0 \\
&- 9a_2(F_1^2 + F_2^2)(F_1 \cos \omega_T t_0 + F_2 \sin \omega_T t_0) \Big] \sin 3kZ \\
&+ \Big\{ \Big[(\alpha_{LT}F_1F_3 - 2\psi_0\omega_T\omega_L F_2F_4) \cos \omega_T t_0 \\
&+ (\alpha_{LT}F_2F_3 + 2\psi_0\omega_T\omega_L F_1F_4) \sin \omega_T t_0 \Big] \cos 2\omega_L t_0 \\
&+ \Big[(\alpha_{LT}F_1F_4 + 2\psi_0\omega_T\omega_L F_2F_3) \cos \omega_T t_0 \\
&+ (\alpha_{LT}F_2F_4 - 2\psi_0\omega_T\omega_L F_1F_3) \sin \omega_T t_0 \Big] \sin 2\omega_L t_0 \Big\} \\
&\times \frac{\sin kZ - 9 \sin 3kZ}{2},
\end{aligned} \tag{65a}$$

$$\frac{\partial^2 V_3}{\partial t_0^2} - c_T^2 \frac{\partial^2 V_3}{\partial Z^2} = 0, \tag{65b}$$

$$\frac{\partial^2 W_3}{\partial t_0^2} - c_L^2 \frac{\partial^2 W_3}{\partial Z^2} = 4\omega_L \left(\frac{\partial F_3}{\partial t_1} \sin 2\omega_L t_0 - \frac{\partial F_4}{\partial t_1} \cos 2\omega_L t_0 \right) \tag{65c}$$

where

$$a_1 = \frac{k^2}{16}\left[\frac{k^2(2\psi_0\omega_T^2 - \alpha_{LT})^2}{2(\omega_L^2 - \omega_T^2)} + \frac{k^2\alpha_{LT}(2\psi_0\omega_T^2 + \alpha_{LT})}{\omega_L^2} - 9\zeta_T \right], \tag{65d}$$

$$a_2 = a_3 + \frac{3}{8}k^2\zeta_T, \tag{65e}$$

$$a_3 = \frac{k^2}{16}\left[\frac{k^2(4\psi_0^2\omega_T^4 - \alpha_{LT}^2)}{2(\omega_L^2 - \omega_T^2)} + \zeta_T^2 \right], \qquad a_4 = a_3 + \frac{k^2}{8}\zeta_T, \tag{65f}$$

and

$$\begin{cases} U_3(Z,0,0,0) = V_3(Z,0,0,0) = W_3(Z,0,0,0) = 0, \\ \dfrac{\partial U_3}{\partial t_0}(Z,0,0,0) = -\dfrac{\partial F_2}{\partial t_2}(0,0,0)\sin kZ, \\ \dfrac{\partial V_3}{\partial t_0}(Z,0,0,0) = 0, \\ \dfrac{\partial W_3}{\partial t_0}(Z,0,0,0) = -\dfrac{\partial W_2}{\partial t_1}(Z,0,0,0) = -\dfrac{\partial F_4}{\partial t_1}(0,0)\cos 2kZ. \end{cases} \tag{65g}$$

From (65) we deduce that $V_3 = 0$, while the solutions for U_3 and W_3 have secular terms containing factors proportional to (59) and to

$$t_0 \cos 2\omega_L t_0 \cos 2kZ \quad \text{and} \quad t_0 \sin 2\omega_L t_0 \cos 2kZ, \tag{66}$$

respectively, unless

$$2\omega_T \frac{\mathrm{d}F_1}{\mathrm{d}t_2} + a_1(F_1^2 + F_2^2)F_2 = 0, \tag{67a}$$

$$2\omega_T \frac{\mathrm{d}F_2}{\mathrm{d}t_2} - a_1(F_1^2 + F_2^2)F_1 = 0, \tag{67b}$$

and

$$\frac{\partial \hat{F}_j}{\partial t_1} = 0 \ (j = 3,4). \tag{68}$$

Solving (67) under the initial conditions (61) yields

$$F_1 = U_0 \cos \omega_{LT}^{(1)} t_2 \quad \text{and} \quad F_2 = U_0 \sin \omega_{LT}^{(1)} t_2, \quad \text{with} \quad \omega_{LT}^{(1)} = \frac{a_1 U_0^2}{2\omega_T}. \tag{69}$$

On the other hand, in view of (68) the solution for W_3 vanishes identically, and combining (68) with (63) implies that the arbitrary functions $F_j = \hat{F}_j(t_2)$ $(j = 3,4)$ satisfy the initial conditions

$$\hat{F}_3(0) = \Gamma U_0^2 \quad \text{and} \quad \hat{F}_4(0) = 0. \tag{70}$$

From the analysis above, (56) and (69) we find that a single transverse wave in the same direction as the initial excitation is produced and, to the leading order, this is

$$U(Z,t;\varepsilon) = \varepsilon U_0 \cos\left[\left(\omega_T - \varepsilon^2 \omega_{LT}^{(1)}\right) t\right] \sin kZ + O(\varepsilon^3). \tag{71}$$

This approximate solution represents a standing wave resulting from the interaction of two waves with the same amplitude $\varepsilon U_0/2$ traveling in opposite directions with the same speed

$$\begin{aligned} v &= \frac{\omega_T - \varepsilon^2 \omega_{LT}^{(1)}}{k} \\ &= c_T - \frac{\varepsilon^2 U_0^2}{32 c_T}\left[\frac{(2\psi_0 k^2 c_T^2 - \alpha_{LT})^2}{2(c_L^2 - c_T^2)} + \frac{\alpha_{LT}(2\psi_0 k^2 c_T^2 + \alpha_{LT})}{c_L^2} - 9\zeta_T\right] \end{aligned} \tag{72}$$

From (72) it is evident that the speed of propagation of the traveling waves whose interaction yields (71) depends on both the wavelength k and amplitude εU_0 of the initial excitation. As an immediate consequence of this, we can conclude that both nonlinearity and powerless internal actions cause wave dispersion.

To determine the leading order approximation of the longitudinal wave we have to proceed our analysis to the next order. To order $O(\varepsilon^4)$, the equation for W_4 reads

$$\begin{aligned} \frac{\partial^2 W_4}{\partial t_0^2} - c_L^2 \frac{\partial^2 W_4}{\partial Z^2} = 2 \Bigg\{ & \Big[k^2(4\psi_2\omega_L^2 + 3\psi_0)\omega_T^2 F_1^2 F_3 - 2k^2\psi_1\omega_T^2\omega_L F_1^2 F_4 \\ & - k^2\beta_{LT} F_2^2 F_3 - 2\omega_L \frac{\mathrm{d}F_4}{\mathrm{d}t_2} \Big] \cos 2\omega_L t_0 \\ & + \Big[k^2(4\psi_2\omega_L^2 + 3\psi_0)\omega_T^2 F_1^2 F_4 + 2k^2\psi_1\omega_T^2\omega_L F_1^2 F_3 \\ & - k^2\beta_{LT} F_2^2 F_4 + 2\omega_L \frac{\mathrm{d}F_3}{\mathrm{d}t_2} \Big] \sin 2\omega_L t_0 \Bigg\} \cos 2kZ + \cdots, \end{aligned} \tag{73}$$

where F_1 and F_2 are as in (69) and, for brevity, only the terms that lead to secular terms in W_4 are reported. Thus, secular terms proportional to those in (66) do not occur in the solution for W_4 if and only if

$$2\omega_L \frac{\mathrm{d}F_4}{\mathrm{d}t_2} - k^2(4\psi_2\omega_L^2 + 3\psi_0)\omega_T^2 F_1^2 F_3 + 2k^2\psi_1\omega_T^2\omega_L F_1^2 F_4 + k^2\beta_{LT} F_2^2 F_3 = 0 \tag{74a}$$

and

$$2\omega_L \frac{\mathrm{d}F_3}{\mathrm{d}t_2} + k^2(4\psi_2\omega_L^2 + 3\psi_0)\omega_T^2 F_1^2 F_4 + 2k^2\psi_1\omega_T^2\omega_L F_1^2 F_3 - k^2\beta_{LT}F_2^2 F_4 = 0. \tag{74b}$$

Solving the ordinary differential equations (74) under the initial conditions (70) yields

$$F_3 = \Gamma U_0^2 \exp\left[-\omega_T^{(1)}\left(t_2 + \frac{\sin 2\omega_{LT}^{(1)} t_2}{2\omega_{LT}^{(1)}}\right)\right]\cos\left[2\left(\omega_{LT}^{(2)} t_2 + \omega_{LT}^{(3)}\frac{\sin 2\omega_{LT}^{(1)} t_2}{2\omega_{LT}^{(1)}}\right)\right], \tag{75a}$$

$$F_4 = \Gamma U_0^2 \exp\left[-\omega_T^{(1)}\left(t_2 + \frac{\sin 2\omega_{LT}^{(1)} t_2}{2\omega_{LT}^{(1)}}\right)\right]\sin\left[2\left(\omega_{LT}^{(2)} t_2 + \omega_{LT}^{(3)}\frac{\sin 2\omega_{LT}^{(1)} t_2}{2\omega_{LT}^{(1)}}\right)\right], \tag{75b}$$

where

$$\omega_T^{(1)} = \frac{k^2\psi_1\omega_T^2 U_0^2}{2}, \tag{75c}$$

$$\omega_{LT}^{(2)} = \frac{k^2 U_0^2}{8\omega_L}[(4\psi_2\omega_L^2 + 3\psi_0)\omega_T^2 - \beta_{LT}], \tag{75d}$$

and

$$\omega_{LT}^{(3)} = \omega_{LT}^{(2)} + \frac{k^2 U_0^2\beta_{LT}}{4\omega_L}. \tag{75e}$$

Finally, from (62b) and (75a) we deduce that, to the leading order, the longitudinal strain is

$$\begin{aligned} W(Z,t;\varepsilon) = \varepsilon^2 U_0^2 \Bigg\{ & \Gamma \exp\left[-\omega_T^{(1)}\left(\varepsilon^2 t + \frac{\sin 2\varepsilon^2\omega_{LT}^{(1)} t}{2\omega_{LT}^{(1)}}\right)\right] \\ & \times \cos\left[2\left(\omega_L t - \varepsilon^2\omega_{LT}^{(2)} t - \omega_{LT}^{(3)}\frac{\sin 2\varepsilon^2\omega_{LT}^{(1)} t}{2\omega_{LT}^{(1)}}\right)\right] + \frac{k^2(2\psi_0\omega_T^2 + \alpha_{LT})}{8\omega_L^2} \\ & - \frac{k^2(2\psi_0\omega_T^2 - \alpha_{LT})}{8(\omega_L^2 - \omega_T^2)}\cos\left[2\left(\omega_T - \varepsilon^2\omega_{LT}^{(1)}\right)t\right]\Bigg\}\cos 2kZ + O(\varepsilon^4). \end{aligned} \tag{76}$$

This approximate solution represents a standing wave which, differently from the transverse wave (71), does not result from the interaction between waves traveling in opposite directions. From (76) we deduce also that if the dispersion modulus ψ_1 is negative (which implies that $\omega_T^{(1)} < 0$), then the amplitude of the standing wave (76) grows fast at large enough times, whence the multiple scale analysis performed here is valid up to times of order $O\left(1/(\varepsilon^2|\omega_T^{(1)}|)\right)$. This shows that powerless internal forces may amplify the amplitude of the longitudinal wave induced by an initial transverse excitation. If $\psi_1 \geq 0$ (so that $\omega_T^{(1)} \geq 0$), then the amplitude of the standing wave (76) is bounded at all times and tends exponentially (in time) to a constant value. Observe that if ψ_1 is negative, then the longitudinal wave (76) can be viewed as the interaction of a breather (the term in (76) that decays exponentially in time) and two counter-propagating traveling waves. However, we conclude that for non-negative values of ψ_1, the approximations obtained here are valid up to times of $O(1/(\varepsilon^3\omega_T))$.

It is worth noting that should the transverse and longitudinal waves produced by a small transverse initial excitation be detected, and the speed of propagation of the induced transverse wave and the peaks $W_{\max}$ and $W_{\min}$ of the induced longitudinal wave be measured at a given point, then from (72) and (76) we could obtain a system of three equations by which we could determine the values of the dispersion parameters ψ_k $(k = 0, 1, 2)$.

8. Traveling wave solutions within the fourth order theory of elasticity

In this section we search for exact wave solutions within the fourth-order theory of elasticity under the assumption that the effects due to the powerless internal actions are small to only play a role at order $O(\epsilon^3)$ (c.f. (25)). This implies that $O(\psi_k/l^2) = O(\epsilon)$ for all $k = 0, 1, 2, 3$. With this assumption Eqs. (45) reduce to

$$\begin{cases} \Psi_{tt} = \dfrac{\partial^2}{\partial Z^2}\left[{c_T}^2\Psi + \alpha_{LT}\Psi W + \beta_{LT}\Psi W^2 + \zeta_T|\Psi|^2\Psi - \psi_0\Psi_t W_t\right], \\ W_{tt} = \dfrac{\partial^2}{\partial Z^2}\left[{c_L}^2\, W + \dfrac{\alpha_{LT}}{2}|\Psi|^2 + \beta_{LT}|\Psi|^2\, W + \xi_L W^2 + \zeta_L W^3 + \psi_0|\Psi_t|^2\right]. \end{cases} \tag{77}$$

In looking for waves traveling in the Z-direction with speed v we introduce the ansatz

$$\Psi = \Psi(s) \quad \text{and} \quad W = W(s), \quad \text{with} \quad s = Z - vt, \tag{78}$$

into (77) to obtain

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}s^2}\left\{\left[\alpha_{LT}W + \beta_{LT}W^2 + \zeta_T|\Psi|^2 - (v^2 - {c_T}^2)\right]\Psi - \psi_0 v^2\Psi' W'\right\} = 0, \\ \dfrac{\mathrm{d}^2}{\mathrm{d}s^2}\left\{\left(\dfrac{\alpha_{LT}}{2} + \beta_{LT}W\right)|\Psi|^2 + \left[\xi_L W + \zeta_L W^2 - (v^2 - {c_L}^2)\right]W + \psi_0 v^2|\Psi'|^2\right\} = 0, \end{cases} \tag{79}$$

where the prime denotes differentiation with respect to s.

Integrating (79) twice yields

$$\begin{cases} \left[\alpha_{LT}W+\beta_{LT}W^2+\zeta_T|\Psi|^2-(v^2-{c_T}^2)\right]\Psi-\psi_0 v^2\Psi' W'=v_1 s+v_0, \\ \left(\dfrac{\alpha_{LT}}{2}+\beta_{LT}W\right)|\Psi|^2+\left[\xi_L W+\zeta_L W^2-(v^2-{c_L}^2)\right]W+\psi_0 v^2|\Psi'|^2=\varsigma_1 s+\varsigma_0, \end{cases} \tag{80}$$

where v_0, v_1, ς_0 and ς_1 are integration constants. Here, we are interested in solutions that are periodic or finite in the limit as s tends to infinity. Therefore, we take $v_1=\varsigma_1=0$. In addition, for the undeformed configuration to be an equilibrium configuration, namely Eqs. (80) are satisfied when the transverse and longitudinal strains vanish identically, also v_0 and ς_0 must vanish.

Next, we insert the Euler representation for the complex strain

$$\Psi(s)=\Omega(s)\exp[i\theta(s)] \tag{81}$$

into (80), factor out the exponential, separate the real and imaginary parts and obtain

$$\begin{cases} [\beta_{LT}W^2+\alpha_{LT}W-(v^2-c_T^2)]\Omega+\zeta_T\Omega^3-\psi_0 v^2\Omega' W'=0, \\ \Omega\theta' W'=0, \\ \left(\dfrac{\alpha_{LT}}{2}+\beta_{LT}W\right)\Omega^2+[\zeta_L W^2+\xi_L W-(v^2-c_L^2)]W+\psi_0 v^2(\Omega'^2+\Omega^2\theta'^2)=0. \end{cases} \tag{82}$$

Obviously, the second equation in (82) is satisfied if (*a*) $\Omega=0$, or (*b*) $W'=0$, or (*c*) $\theta'=0$. One can readily prove that if the amplitude of the complex strain vanishes, then no traveling wave can propagate. Instead, if the volume strain is constant, then only Carroll-type wave solutions as those analyzed in Section 6 exist. Therefore, the only case worth to be investigated is the one in which the argument of the complex strain is constant. From a physical point of view, this means that should traveling transverse waves exist, they would be linearly polarized. With $\theta'=0$, Eqs. (82) reduce to

$$\begin{cases} [\beta_{LT}W^2+\alpha_{LT}W-(v^2-c_T^2)]\Omega+\zeta_T\Omega^3-\psi_0 v^2\Omega' W'=0, \\ \left(\dfrac{\alpha_{LT}}{2}+\beta_{LT}W\right)\Omega^2+[\zeta_L W^2+\xi_L W-(v^2-c_L^2)]W+\psi_0 v^2\Omega'^2=0. \end{cases} \tag{83}$$

Observe that in the absence of a powerless contribution to the stress tensor (i.e. $\psi_0=0$), Eqs. (83) form an algebraic system and thus no traveling wave solution of (77) exists.

Multiplying the first equation in (79) by Ω', the second one by W' and summing term by term yield the first integral

$$\zeta_T\Omega^4+2f(W)\Omega^2+g(W)=\mathcal{E}, \tag{84a}$$

where

$$f(W)=\beta_{LT}W^2+\alpha_{LT}W-(v^2-{c_T}^2), \tag{84b}$$

$$g(W)=\zeta_L W^4+\frac{4}{3}\xi_L W^3-2(v^2-{c_L}^2)W^2 \tag{84c}$$

and $\mathcal{E}$ is an integration constant. Again, for the undeformed configuration to be an equilibrium configuration, the integration constant $\mathcal{E}$ must vanish.

Should $\Delta(W)=f^2(W)-\zeta_T g(W)$ be non-negative, Eq. (84) would admit the real roots

$$\Omega_{\pm}^2=\frac{-f(W)\pm\sqrt{f^2(W)-\zeta_T g(W)}}{\zeta_T}\equiv\mathscr{F}_{\pm}(W). \tag{85}$$

Clearly, it makes sense to consider only positive roots for Ω^2.

Next, providing that $\mathscr{F}_+(W)$ and/or $\mathscr{F}_-(W)$ are positive, we can express W'^2 in terms of the longitudinal strain W. Indeed, multiplying the first equation in (83) by Ω and making use of (85) we obtain

$$W'^2=\pm\frac{2}{\psi_0 v^2}\frac{\mathscr{F}_{\pm}(W)}{\dfrac{\mathrm{d}\mathscr{F}_{\pm}}{\mathrm{d}W}(W)}\sqrt{\Delta(W)}\equiv\frac{c_T^2}{v^2\psi_0}\mathscr{G}_{\pm}(W). \tag{86}$$

From this equation one deduces that accounting for powerless contributions to the Cauchy stress tensor might allow the existence of traveling wave solutions. Depending on the values of the material parameters (44b)–(44i) and the speed of propagation v, the functions $\mathscr{G}_+(W)$ and/or $\mathscr{G}_-(W)$ may be positive in a finite sub-interval of $]-1,+\infty[$ and have zeros with multiplicity greater or equal to one. Depending on the multiplicity of the zeros, a qualitative analysis may reveal the existence of kink, solitary or periodic wave solutions.

8.1. Periodic waves in carbon black-reinforced natural rubber

Since performing a qualitative study of Eqs. (84) and (86) is very complicated in the general case because of the presence of several material constants, for simplicity, we here limit our attention to a special material: the carbon black-reinforced natural rubber whose strain energy density is modeled by the response function (29).

Within the weakly nonlinear theory of elasticity of the fourth order, the strain energy (29) approximates as (44), with

$$\begin{aligned} c_T &= 25.23\,\mathrm{m/s}, \quad c_L = 826.24\,\mathrm{m/s}, \quad \alpha_{LT} = -6.67\cdot 10^2\,\mathrm{m^2/s^2},\\ \xi_L &= -3.42\cdot 10^5\,\mathrm{m^2/s^2}, \quad \zeta_T = 85.45\,\mathrm{m^2/s^2},\\ \beta_{LT} &= 8.31\cdot 10^2\,\mathrm{m^2/s^2}, \quad \zeta_L = 3.43\cdot 10^5\,\mathrm{m^2/s^2}. \end{aligned} \tag{87}$$

The values of these elastic moduli have been determined from (29) and (44b) to (44i), considering that the mass density ρ_0 of the samples of carbon black-reinforced natural rubber used by Cantournet et al. (2014) to determine the best fit parameters in (29) is $1.1\cdot 10^3\ \mathrm{kg/m^3}$.

From Eqs. (85) and (86), with the elastic moduli as in (87), one can check that only supersonic traveling waves (i.e., waves whose speed of propagation v exceeds in modulus the speed of propagation of transverse waves in the linear regime c_T) are allowed. Depending on the sign of the dispersion parameter ψ_0, the supersonic traveling waves allowed are accompanied by volume contraction or expansion. Specifically, we distinguish the following two cases (Fig. 2).

- If $\psi_0 > 0$ and $v^2 > c_T^2$, then there exists $W_S \in\,]-1, 0[$ such that $\mathscr{G}_-$ is positive in $]W_S, 0[$, with W_S and 0 being simple zeros of $\mathscr{G}_-$. Thus, a qualitative analysis of (86) reveals that the longitudinal wave is periodic and given by

$$W = \begin{cases} \mathcal{H}^{-1}\left(\dfrac{s-s_0}{\sqrt{|\psi_0|}} - 2n\kappa_S\right) & \text{if} \quad \dfrac{s-s_0}{\sqrt{|\psi_0|}} \in [(2n-1)\kappa_S, 2n\kappa_S],\\ \mathcal{H}^{-1}\left(2n\kappa_S - \dfrac{s-s_0}{\sqrt{|\psi_0|}}\right) & \text{if} \quad \dfrac{s-s_0}{\sqrt{|\psi_0|}} \in [2n\kappa_S, (2n+1)\kappa_S], \end{cases} \tag{88a}$$

  where

$$\mathcal{H} : \varpi \in [W_S, 0] \mapsto -\int_{\varpi}^{0} \frac{\mathrm{d}\varsigma}{\sqrt{\mathscr{G}_-(\varsigma)}} \in [-\kappa_S, 0], \tag{88b}$$

$$\kappa_S = \frac{c_T}{|v|}\int_{W_S}^{0} \frac{\mathrm{d}\varsigma}{\sqrt{\mathscr{G}_-(\varsigma)}}, \tag{88c}$$

  $n \in \mathbb{Z}$ and s_0 is an integration constant.
  On the other hand, in view of (85), the modulus of the complex transverse strain is in the form

$$\Omega = \sqrt{\mathscr{F}_-(W)}, \tag{89}$$

  with W as in (88a).
- If $\psi_0 < 0$ and $v^2 > c_T^2$, then there exists $W_E > 0$ such that $\mathscr{G}_-/\psi_0$ is positive in $]0, W_E[$, with 0 and W_E being simple zeros of $\mathscr{G}_-$. Consequently, a standard qualitative analysis of (86) shows that the longitudinal wave is periodic and given by

$$W = \begin{cases} \mathcal{L}^{-1}\left(\dfrac{s-s_0}{\sqrt{|\psi_0|}} - 2n\kappa_E\right) & \text{if} \quad \dfrac{s-s_0}{\sqrt{|\psi_0|}} \in [2n\kappa_E, (2n+1)\kappa_E],\\ \mathcal{L}^{-1}\left(2n\kappa_E - \dfrac{s-s_0}{\sqrt{|\psi_0|}}\right) & \text{if} \quad \dfrac{s-s_0}{\sqrt{|\psi_0|}} \in [(2n-1)\kappa_E, 2n\kappa_E], \end{cases} \tag{90a}$$

  where

$$\mathcal{L} : \varpi \in [0, W_E] \mapsto \int_{0}^{\varpi} \frac{\mathrm{d}\varsigma}{\sqrt{|\mathscr{G}_-(\varsigma)|}} \in [0, \kappa_E], \tag{90b}$$

$$\kappa_E = \frac{c_T}{|v|}\int_{0}^{W_E} \frac{\mathrm{d}\varsigma}{\sqrt{|\mathscr{G}_-(\varsigma)|}}, \tag{90c}$$

  and, as before, $n \in \mathbb{Z}$ and s_0 is an integration constant.
  Again, from (85) the modulus of the complex transverse strain is given by (89) with W as in (90a).

In any other case different from those reported above, no solution of (86) represents a physically admissible traveling wave.

In Fig. 2 it is evident that irrespective of the sign of the dispersion parameter ψ_0, both W_S and W_E tend to zero as $v^2 \to c_T^2$. This deduction combined with (88a), (90a) and (89) implies that the waveforms of longitudinal and transverse waves tend uniformly to zero. Thus, sonic traveling waves (whose speed of propagation is equal to c_T) do not propagate in carbon black-reinforced natural rubber.

Summarizing, if ψ_0 is positive (respectively, if ψ_0 is negative), the longitudinal traveling wave solution (88a) (respectively, (90a)) is smooth and oscillates between two equilibria: the undeformed configuration and a shrunk (respectively, expanded) configuration corresponding to $W = W_S \in\,]-1, 0[$ (respectively, $W = W_E > 0$). The modulus of the transverse wave (89) fails to be continuously differentiable at its maximum points (Fig. 3). In fact, at the peaks of the transverse wave Ω' has a jump discontinuity with magnitude equal to

$$\sqrt{\left|\frac{1}{\psi_0 v^2 \zeta_T}\left(2f(W_k)\frac{\mathrm{d}f}{\mathrm{d}W}(W_k) - \zeta_T\frac{\mathrm{d}g}{\mathrm{d}W}(W_k)\right)\right|}, \tag{91}$$

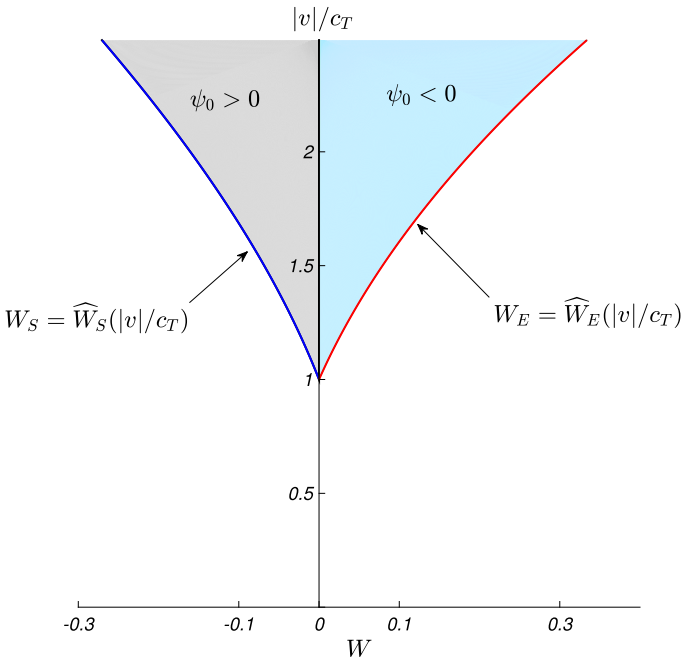


Fig. 2. Regions in the W $|v|/c_T$-plane in which traveling waves with periodic waveform can propagate in carbon black-reinforced natural rubber. The cyan region is in correspondence with negative values of the dispersion parameter ψ_0, the gray one with positive values of ψ_0. The blue, red and (vertical) black curves are the zero loci of $\mathscr{G}_-$.

where $k = S$ or $k = E$ depending on whether the longitudinal wave oscillates between the undeformed and a shrunk configurations or between the undeformed and an expanded configurations. From a mathematical point of view, this means that the wave solutions determined in this section represent generalized solutions of (77).

The longitudinal and transverse waves oscillate with the same wavelength

$$\lambda_S = \kappa_S \sqrt{\psi_0} \quad \text{(respectively, } \lambda_E = \kappa_E \sqrt{|\psi_0|}\text{)}. \tag{92}$$

These formulas can be used to determine the value of the material parameter ψ_0. In fact, should periodic longitudinal waves be detected in carbon black-reinforced natural rubber and their wavelengths be measured, then ψ_0 could be calculated from (92). More precisely, ψ_0 would be positive if the longitudinal wave oscillated between the undeformed and a shrunk configurations. It would be negative if the longitudinal wave oscillated between the undeformed and an expanded configurations.

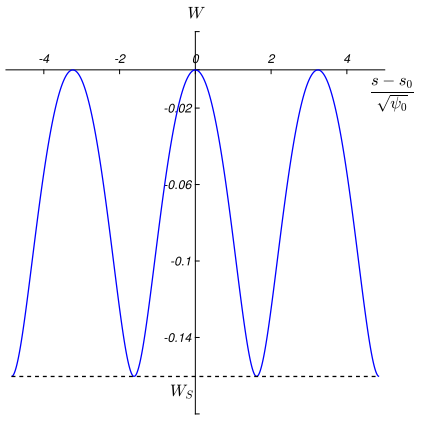
Finally, we point out that the dimensionless wavelengths (κ_S or κ_E) and the amplitudes of the longitudinal and transverse waves depend on the speed of propagation of the waves themselves as displayed in Fig. 4.

9. Concluding remarks

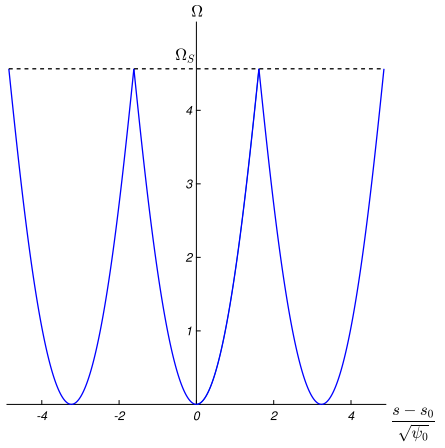
In this paper, we have analyzed the propagation of longitudinal and transverse waves in compressible hyperelastic solids when powerless internal forces (which have thus far been ignored) are taken into account. Starting from the most general model for the strain energy density of an isotropic hyperelastic material and the most general model for the powerless part of the stress tensor within the framework of simple materials of the second grade, we have derived the equations of motion in the fully nonlinear setting and found some exact solutions. Within the weakly non linear theory of elasticity of the fourth order, we have instead determined some approximate solutions.

In particular, we have shown that powerless internal forces do not affect the propagation of transverse waves in solids subjected to a uniform contraction or expansion.

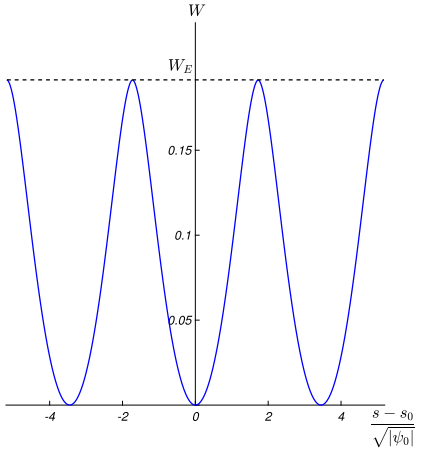
Despite the complexity of the governing equations in the fully nonlinear regime, we analyzed the interaction between the transverse and longitudinal waves excited by a small but finite purely transverse initial excitation by using the method of multiple scales. This method is preferable to the standard perturbation analysis as it prevents the occurrence of secular terms. From this analysis it emerges that the powerless internal forces play a role in the intricate mechanism of interaction between longitudinal and



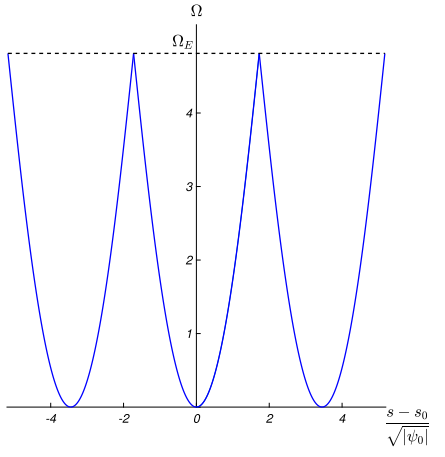
(a) Longitudinal wave allowed when ψ_0 is positive.



(b) Transverse wave allowed when ψ_0 is positive.



(c) Longitudinal wave allowed when ψ_0 is negative.



(d) Transverse wave allowed when ψ_0 is negative.

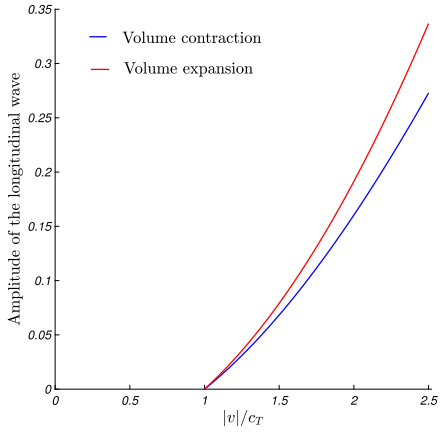
Fig. 3. Profiles of longitudinal and transverse waves that may propagate in carbon black-reinforced natural rubber in the presence of powerless contributions to the stress tensor. Here, for illustration, we have taken the speed of propagation $v = 2c_T = 50.46$ m/s.

transverse waves as, in conjunction with nonlinearity, it causes wave dispersion and, for some values of the dispersion parameters, may amplify the amplitude of the longitudinal wave produced by an initial purely transverse excitation.

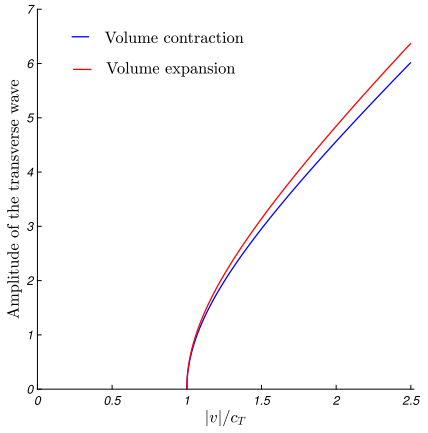
Within the fourth order weakly nonlinear theory of elasticity, we have investigated the existence of traveling wave solutions, and proven their existence when a powerless part of the stress tensor is included in the constitutive assumptions on the mechanical response of the material. On the contrary, in the absence of contributions due to the powerless internal forces to the stress tensor, traveling wave cannot propagate in a compressible hyperelastic solid (c.f. (83)). For illustration, for carbon black-reinforced natural rubber whose strain energy is modeled as in (29) we have shown the existence of small amplitude periodic longitudinal and transverse waves which propagate with the same wavelength.

CRediT authorship contribution statement

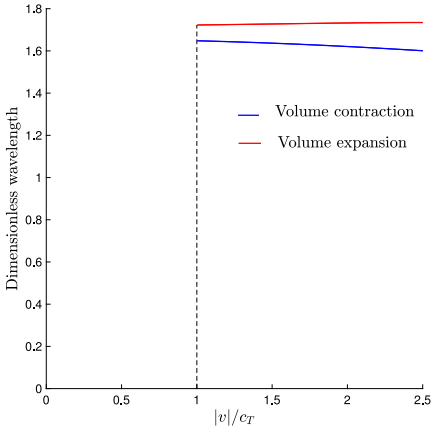
Ada Amendola: Writing – review & editing, Writing – original draft, Validation, Investigation, Funding acquisition, Formal analysis. **Julia de Castro Motta:** Writing – review & editing, Writing – original draft, Validation, Investigation, Formal analysis. **Luigi Vergori:** Writing – review & editing, Writing – original draft, Validation, Investigation, Funding acquisition, Formal analysis.



(a) Amplitudes of the longitudinal waves.



(b) Amplitudes of the transverse waves.



(c) Dimensionless wavelengths ($\lambda_S/\sqrt{\psi_0}$ and $\lambda_E/\sqrt{|\psi_0|}$) of the periodic waves.

Fig. 4. Amplitudes and dimensionless wavelengths of the longitudinal and transverse waves vs the dimensionless speed of propagation $|v|/c_T$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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