Received: 25 August 2024 | Revised: 22 October 2024 | Accepted: 25 October 2024

DOI: 10.1002/zamm.202400869

ORIGINAL PAPER

On the linearized Eringen–Cattaneo–Christov–Straughan model: Some qualitative properties of solutions

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Funding information
Italian National Group of Mathematical Physics (GNFM–INdAM), Grant/Award Numbers: 2022P5R22A, D53D23003890006; National Recovery and Resilience Plan (NRRP), Grant/Award Numbers: P2022PE8BT, D53D23018440001

In this paper the Eringen–Cattaneo–Christov–Straughan linearized model is investigated, studying for it in particular: (i) the well-posedness in Hadamard's sense of the related initial-boundary value problem, by resorting to the semi-group theory; and (ii) the type of decay the solutions are subject to. This study contributes to strengthening the theoretical basis of a recent model capable of describing thermoporoacoustic phenomena.

1 | INTRODUCTION

The aim of this work is to highlight several qualitative aspects of the solutions of a coupled problem recently described in ref. [1, §2.4], and related to the so-called (linearized) Eringen–Cattaneo–Christov–Straughan (ECCS) model, able to describe thermoporoacoustic phenomena. The model studied originates from Eringen (see refs. [2, 3]) and it is useful to remember that, about twenty years ago, it was thoroughly investigated within the context of linear theory by several authors; in this regard, reference should be made to the group of works by Chiriţă (see refs. [4, 5]), by Galeş (see refs. [6–11]), and by Quintanilla (see refs. [12–17], directly involving swelling porous elasticity, or ref. [18], highlighting similarities with Eringen's model). In more recent times, we also record wave propagation issues in the linear, isothermal case (see, for example, refs. [19, 20]).

As it is well known, Fourier's law predicts that even infinitesimal heat disturbances are able to propagate with an infinite speed through the conducting medium; in order to suppress such paradox, a large number of so-called "non-Fourier" heat conduction theories have been proposed (and explored) over the years. As specified in ref. [1], in this direction and with

respect to the above model, Straughan [21, §7] included the second sound effects, that is, activated thermal waves, replacing the Fourier heat flux relation with the Cattaneo–Christov one (refer to ref. [22, §3.1.2] for a brief description, and references therein for further information):

$$\tau[\mathbf{q}_t + (\mathbf{v}\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\mathbf{v} + (\nabla\cdot\mathbf{v})\mathbf{q}] + \mathbf{q} = -\bar{k}\nabla\vartheta, \tag{1}$$

where $\mathbf{v}$ stands for the velocity vector and $\mathbf{q}$ for the heat flux vector, while ϑ is the absolute temperature, $\bar{k}$ the thermal conductivity, and τ the Cattaneo relaxation time, all referring to the fluid. Our analysis starts, in detail, from the linearized model described by sys. (14) (see ref. [1], p. 240), suitably adapted to a three-dimensional domain identifying a rigid, stationary porous matrix permeated – for us – by a perfect gas. It is worth remembering that numerous and in-depth studies exist, that involve in particular wave propagation phenomena, concerning the fields of poroacoustics and thermoporoacoustics. Some examples in this sense – facing both linear and, mainly, nonlinear theories – include, but are not limited to, the works by Jordan (see, for example, refs. [23–25]) and Straughan (see, for example, refs. [26–28]). Different approaches exist to the problem, even with regard to the modelling of heat exchange phenomena, which is why for a deeper bibliographic review of the topic we refer the reader to the works cited therein. On the other hand, it should be remembered that well-posedness and qualitative properties of solutions to elasticity and thermoelasticity problems involving porous media are also, classically, topics of great appeal. Here we limit ourselves to recalling some works by Chiriţă and Scalia (see ref. [29] where, in the context of linear theory, the spatial and temporal behaviour of the solutions is investigated), by Ieşan (we just cite ref. [30], in which uniqueness, reciprocal and variational results are given in the context of the linear theory, along with an acceleration waves analysis), again by Quintanilla [31, 32], or even by Svanadze (see ref. [33], to make a reference to the concept of multi-porosity, which however goes beyond the purposes of the present work).

The minimal part, just recalled as an example, of the immense bibliography on the subject makes us understand the interest existing on the problem, both from an engineering point of view (refer in this regard to ref. [26, §1] or even ref. [27, §1]) and a more strictly mathematical one; the present work is evidently directed towards the latter – in fact, it addresses and deepens mathematical aspects of the linearized ECCS model to which, in ref. [1], the singular-surface theory has been applied.

We aim first to show that, under appropriate hypotheses, the model under investigation is well posed (*parfaitement bien posé*) in Hadamard's interpretation, proving existence, uniqueness, and continuous dependence of solution through the semigroup theory for linear operators. In order to achieve these goals, sys. (2) (not previously studied from this point of view) needs preliminary manipulation in order to be expressed in the alternative forms (4) and (8). Such manipulation allows us to describe our problem as a Cauchy problem in a suitable Hilbert space. Subsequently, we are able to detect the type of decay that characterizes the solutions. In particular, we resort to the Routh–Hurwitz rule to clarify the strongly stable character of the solutions to the problem. Finally, the energy method allows us to check the exponential stability of the solutions.

The outline of the paper is the following: In Section 2 we give a detailed description of the mathematical model investigated. An existence result – along with related findings of uniqueness and continued dependence – is given in Section 3, while Section 4 is devoted to the study of the decay of solutions. In conclusion, some final remarks (Section 5) are given.

2 | DEFINITION OF THE PROBLEM

According to ref. [1], sys. (14), let us consider a bounded domain $\mathcal{B}$ of $\mathbb{R}^3$ whose boundary $\partial\mathcal{B}$ is smooth enough to apply the divergence theorem (say, $\partial\mathcal{B}$ is of class C^1), and write the system of partial differential equations defining – for the purpose, in the absence of any body force – the (three-dimensional) model:

$$\begin{cases} P_{tt} + P_t - \Delta P = \Theta_{tt} + \Theta_t + A\Delta\Theta \\ \beta\gamma\Theta_{tt} + \gamma\Theta_t - m^2\Delta\Theta = (\gamma-1)(\beta P_{tt} + P_t) \end{cases} \tag{2}$$

where Δ stands for the Laplace operator, $A = \sigma_0\vartheta_0/p_0$ and $\beta = \tau_0\delta_D$.

We recall that such system is given in a completely dimensionless form, and that $P = (p - p_0)/p_0$ and $\Theta = (\vartheta - \vartheta_0)/\vartheta_0$ are the dimensionless acoustic pressure and excess temperature variables, respectively, being p the thermodynamic pressure and ϑ the absolute temperature of the gas saturating the porous matrix; the "zero" subscript identifies the corre-

sponding uniform ambient state value. Moreover, as in ref. [1], σ_0 identifies the so-called "Eringen coefficient", τ_0 is the relaxation time, and $\delta_D = \nu_0\chi/\mathbb{P}$, where ν_0 is the kinematic viscosity of the gas while $\chi \in (0,1)$ and $\mathbb{P}(>0)$ are the porosity and permeability of the porous matrix, respectively. In addition, $m = b_0^{-1}\sqrt{\gamma\kappa_0\delta_D}$, where b_0 is the isothermal sound speed in the gas and κ_0 is the thermal diffusivity of the gas. At this point, it is useful to make some clarifications/assumptions:

1. γ is defined as the ratio c_p/c_v, and so we assume $\gamma > 1$, being c_p, c_v the specific heats at constant pressure and volume, respectively;
2. as for $\beta = \tau_0\delta_D$, given the above definitions, it is reasonable to take $\beta > 0$;
3. a different situation occurs for A instead since – we recall – in ref. [1], the analysis is performed for $\sigma_0 \geq \sigma_0^*$, where $\sigma_0^* = -p_0/\vartheta_0$, that is, A could also attain negative values[1]. For the main line of investigation in this work, however, $A > 0$ will be assumed.

In summary, we are going to consider here the following hypotheses to be valid:

$$\gamma > 1 \qquad A > 0 \qquad \beta > 0 \tag{3}$$

Isolating $\Delta\Theta$ from Equation $(2)_2$, through simple manipulations sys. (2) can be rewritten as

$$\begin{cases} \left[1 + \dfrac{A}{m^2}\beta(\gamma-1)\right]P_{tt} - \left(1 + \dfrac{A}{m^2}\beta\gamma\right)\Theta_{tt} = \Delta P - \left[1 + \dfrac{A}{m^2}(\gamma-1)\right]P_t + \left(1 + \dfrac{A}{m^2}\gamma\right)\Theta_t \\ -(\gamma-1)\beta P_{tt} + \beta\gamma\Theta_{tt} = m^2\Delta\Theta + (\gamma-1)P_t - \gamma\Theta_t \end{cases} \tag{4}$$

Defining

$$B = \begin{pmatrix} 1 + \dfrac{A}{m^2}\beta(\gamma-1) & -\left(1 + \dfrac{A}{m^2}\beta\gamma\right) \\ -(\gamma-1)\beta & \beta\gamma \end{pmatrix} \tag{5}$$

and

$$C = \begin{pmatrix} -\left[1 + \dfrac{A}{m^2}(\gamma-1)\right] & 1 + \dfrac{A}{m^2}\gamma \\ \gamma - 1 & -\gamma \end{pmatrix} \tag{6}$$

sys. (4) can, in turn, be expressed as follows

$$B\begin{pmatrix} P_{tt} \\ \Theta_{tt} \end{pmatrix} = \begin{pmatrix} \Delta P \\ m^2\Delta\Theta \end{pmatrix} + C\begin{pmatrix} P_t \\ \Theta_t \end{pmatrix} \tag{7}$$

where we note that $\det B = \beta = \tau_0\delta_D$, taken here greater than zero (see (3)): so, B is not singular (we also record that $\det C = 1$). Said $L = (l_{ij}) = B^{-1}$, from (7) we deduce

$$\begin{aligned} \{P_{tt}, \Theta_{tt}\} = &\left[\Delta P - \left(1 + \frac{A}{m^2}(\gamma-1)\right)P_t + \left(1 + \frac{A}{m^2}\gamma\right)\Theta_t\right]\{l_{11}, l_{21}\} \\ &+\left[m^2\Delta\Theta + (\gamma-1)P_t - \gamma\Theta_t\right]\{l_{12}, l_{22}\} \end{aligned} \tag{8}$$

In order to completely define a well-posed initial-boundary value problem, to the system of partial differential equations (2) – and subsequent manipulations – we add the following:

[1] In this case it would turn out, in particular, $A \geq -1$. We refer the reader to ref. [1, §2.3], p. 240, where the question *"Is σ_0^* an admissible value of σ_0?"* is formulated.

1. (homogeneous) boundary conditions

$$P(\mathbf{x},t)=\Theta(\mathbf{x},t)=0, \qquad \mathbf{x}\in\partial\mathcal{B},\ t\in\mathbb{R}^+ \tag{9}$$

2. and initial conditions

$$\begin{aligned} &P(\mathbf{x},0)=P^0(\mathbf{x}), \quad P_t(\mathbf{x},0)=Q^0(\mathbf{x}),\\ &\Theta(\mathbf{x},0)=\Theta^0(\mathbf{x}), \quad \Theta_t(\mathbf{x},0)=\eta^0(\mathbf{x}), \end{aligned} \qquad \mathbf{x}\in\mathcal{B} \tag{10}$$

As for boundary conditions (9), we recall that setting (dimensionless) acoustic pressure P and excess temperature Θ equal to zero means to set, respectively, $p=p_0$ and $\vartheta=\vartheta_0$ along the boundary $\partial\mathcal{B}$.

With reference to the problem (2), (9), (10) it is not difficult to prove the validity of the following equality

$$E(t)+\int_0^t D(s)ds=E(0), \tag{11}$$

where

$$\begin{aligned} E(t)=\frac{1}{2}\int_{\mathcal{B}}\Big\{&\beta(\gamma-1)\Big[1+\frac{A}{m^2}\beta(\gamma-1)\Big]P_t^2+\beta\gamma\Big(1+\frac{A}{m^2}\beta\gamma\Big)\Theta_t^2\\ &-2\beta(\gamma-1)\Big(1+\frac{A}{m^2}\beta\gamma\Big)P_t\Theta_t+\beta(\gamma-1)|\nabla P|^2+m^2\Big(1+\frac{A}{m^2}\beta\gamma\Big)|\nabla\Theta|^2\Big\}dv \end{aligned} \tag{12}$$

and

$$\begin{aligned} D(t)=\int_{\mathcal{B}}\Big\{&\beta(\gamma-1)\Big[1+\frac{A}{m^2}(\gamma-1)\Big]P_t^2+\gamma\Big(1+\frac{A}{m^2}\beta\gamma\Big)\Theta_t^2\\ &-(\gamma-1)\Big(1+\beta+2\frac{A}{m^2}\beta\gamma\Big)P_t\Theta_t\Big\}dv \end{aligned} \tag{13}$$

At this point, we must emphasize what follows. Assumptions (3) are sufficient to guarantee that the function $E(t)$ given in (12) is always non-negative. In fact, in it we can recognize the sum of two non-negative terms (in ∇P and $\nabla\Theta$) plus a positive definite quadratic form in the variables P_t and Θ_t, whose associated matrix has in fact determinant $\beta^2(\gamma-1)\left(1+A\beta\gamma/m^2\right)$. On the contrary, as far as function $D(t)$ given in (13) is concerned, we record that assumptions (3) are *no longer sufficient* to guarantee its non-negativity. In addition, it would be enough to impose the following condition:

$$4A\beta^2\gamma+m^2\Big[(1+\beta)^2-\gamma(1-\beta)^2\Big]>0 \tag{14}$$

Condition (14) determines a relation between the coefficients of the problem to guarantee the dissipative character of the energy provided by the function (12). Evidently, it introduces a further restriction among the constitutive parameters. We also note that, under assumptions (3) and (14), the equality (11) implies the stability of the solutions; later, we will discuss about the exponential stability.

3 | EXISTENCE, UNIQUENESS, AND CONTINUOUS DEPENDENCE OF THE SOLUTION

In this Section, we provide an existence (as well as, uniqueness and continuous dependence) result for the solution of the problem given by Equations (8)–(10). Let us assume the hypotheses (3) always valid, that is, the function $E(t)$ given in (12) non-negative. Furthermore, let us consider the Hilbert space

$$\mathcal{H}=W_0^{1,2}(\mathcal{B})\times L^2(\mathcal{B})\times W_0^{1,2}(\mathcal{B})\times L^2(\mathcal{B}), \tag{15}$$

where $W_0^{1,2}(\mathcal{B})$ and $L^2(\mathcal{B})$ are the usual Sobolev spaces. The elements in this space can be written as $\mathcal{U} = (P, Q, \Theta, \eta)$, and we consider in $\mathcal{H}$ the inner product associated with the norm[2]:

$$\begin{aligned}\|\mathcal{U}\|^2 = \frac{1}{2}\int_{\mathcal{B}}\Big\{&\beta(\gamma-1)\Big[1+\frac{A}{m^2}\beta(\gamma-1)\Big]Q^2+\beta\gamma\Big(1+\frac{A}{m^2}\beta\gamma\Big)\eta^2\\&-2\beta(\gamma-1)\Big(1+\frac{A}{m^2}\beta\gamma\Big)Q\eta+\beta(\gamma-1)|\nabla P|^2+m^2\Big(1+\frac{A}{m^2}\beta\gamma\Big)|\nabla\Theta|^2\Big\}dv,\end{aligned} \tag{16}$$

equivalent to the usual one in the Hilbert space $\mathcal{H}$ in view of our assumptions.

Now, we define the linear operator $\mathcal{A}$ such that

$$\mathcal{A}\begin{pmatrix}P\\Q\\\Theta\\\eta\end{pmatrix}=\begin{pmatrix}Q\\M_1\\\eta\\M_2\end{pmatrix}, \tag{17}$$

where, for $j = 1, 2$,

$$M_j = \Big[\Delta P-\Big(1+\frac{A}{m^2}(\gamma-1)\Big)Q+\Big(1+\frac{A}{m^2}\gamma\Big)\eta\Big]l_{j1}+\big[m^2\Delta\Theta+(\gamma-1)Q-\gamma\eta\big]l_{j2} \tag{18}$$

Our problem can be written as

$$\frac{d}{dt}\mathcal{U}=\mathcal{A}\mathcal{U},\quad \mathcal{U}(0)=\mathcal{U}^0=\big(P^0,Q^0,\Theta^0,\eta^0\big) \tag{19}$$

It is worth noting that the domain of the operator $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$, is given by the elements of $\mathcal{H}$ such that $Q, \eta \in W_0^{1,2}(\mathcal{B})$ and $P, \Theta \in W_0^{2,2}(\mathcal{B})$; therefore, $\mathcal{D}(\mathcal{A})$ is a dense subspace of $\mathcal{H}$.

In view of the field equations and boundary conditions, and using the divergence theorem, for every $\mathcal{U} \in \mathcal{D}(\mathcal{A})$ we see that

$$\begin{aligned}\langle\mathcal{A}\mathcal{U},\,\mathcal{U}\rangle = -\frac{1}{2}\int_{\mathcal{B}}\Big\{&\beta(\gamma-1)\Big[1+\frac{A}{m^2}(\gamma-1)\Big]Q^2+\gamma\Big(1+\frac{A}{m^2}\beta\gamma\Big)\eta^2\\&-(\gamma-1)\Big(1+\beta+2\frac{A}{m^2}\beta\gamma\Big)Q\eta\Big\}dv\end{aligned} \tag{20}$$

Given the definition of the norm (16), we know that there exists a positive constant C such that

$$\langle\mathcal{A}\mathcal{U},\,\mathcal{U}\rangle\le C\|\mathcal{U}\|^2, \tag{21}$$

for every $\mathcal{U} \in \mathcal{D}(\mathcal{A})$. In order to prove that $\mathcal{A}$ defines a quasi-contractive semigroup, it is sufficient to show that zero belongs to the resolvent of the operator $\mathcal{A}$. To do this, let $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ and prove that an element $\mathcal{U} \in \mathcal{D}(\mathcal{A})$ exists such that

$$\mathcal{A}\mathcal{U}=\mathcal{F} \tag{22}$$

[2] In the previous Section, we assumed conditions (3). However, the inner product that is going to be given remains well-defined also in the relaxed case: $\gamma > 1$, $\beta > 0$ and $m^2 + A\beta\gamma > 0$. Following the nomenclature given in ref. [1], it is interesting to note that the restriction given by the third inequality is equivalent to $\sigma_0 > \left(a_0^2/c_0^2\right)\sigma_0^*$, where a_0 stands for the purely thermal waves speed, and $c_0 = b_0\sqrt{\gamma}$. In recording that a_0 may be greater than, less than or even equal to c_0 (see ref. [1], p. 240) and that $a_0 = c_0$ identifies one of the critical values (see ref. [1], p. 249) for the problem studied, we note that our inner product can remain well-defined even for negative values of A, that is, relaxing the hypothesis $(3)_2$.

We are led to the system:

$$\begin{cases} Q = f_1 \\ l_{11}\left[\Delta P - \left(1 + \frac{A}{m^2}(\gamma - 1)\right)Q + \left(1 + \frac{A}{m^2}\gamma\right)\eta\right] + l_{12}\left[m^2\Delta\Theta + (\gamma - 1)Q - \gamma\eta\right] = f_2 \\ \eta = f_3 \\ l_{21}\left[\Delta P - \left(1 + \frac{A}{m^2}(\gamma - 1)\right)Q + \left(1 + \frac{A}{m^2}\gamma\right)\eta\right] + l_{22}\left[m^2\Delta\Theta + (\gamma - 1)Q - \gamma\eta\right] = f_4 \end{cases} \tag{23}$$

from which, on the one hand, we obtain Q and η; on the other, the system becomes

$$\begin{cases} l_{11}\Delta P + l_{12}m^2\Delta\Theta = F_1 \\ l_{21}\Delta P + l_{22}m^2\Delta\Theta = F_2 \end{cases} \tag{24}$$

where

$$F_j = f_{2j} - l_{j1}\left[-\left(1 + \frac{A}{m^2}(\gamma - 1)\right)f_1 + \left(1 + \frac{A}{m^2}\gamma\right)f_3\right] - l_{j2}[(\gamma - 1)f_1 - \gamma f_3], \tag{25}$$

for $j = 1, 2$, and where f_{2j} has to be meant f_2 for $j = 1$ and f_4 for $j = 2$. Since $L = \left(l_{ij}\right) = B^{-1}$, sys. (24) admits a solution in $W_0^{1,2}(\mathcal{B}) \cap W^{2,2}(\mathcal{B})$ and

$$\|\mathcal{U}\| \leq K\|\mathcal{F}\|, \tag{26}$$

where K is a constant independent of the data. Thus, in view of the Lumer–Phillips corollary to the Hille–Yosida theorem, we can guarantee that $\mathcal{A}$ generates a quasi-contractive semigroup in $\mathcal{H}$ (see, for example, ref. [34]).

Theorem 1. *Let the assumptions (3) be valid. For every $\mathcal{U}^0 = \left(P^0, Q^0, \Theta^0, \eta^0\right) \in \mathcal{D}(\mathcal{A})$, there exists a unique solution to the problem determined by Equations (8)–(10). In fact*

$$\mathcal{U}(t) \in C^1([0, t], \mathcal{H}) \cap C^0([0, t], \mathcal{D}(\mathcal{A})) \tag{27}$$

Remark 1. It is worth noting that the existence of a semigroup also ensures the continuous dependence of the solution with respect to initial data and supply terms (if assumed). We can therefore conclude that our problem is well-posed in the sense of Hadamard.

4 | STUDY OF THE DECAY OF SOLUTIONS

The aim of this Section is to identify sufficient conditions to guarantee the decay of the solutions and, possibly, determine the kind of decay. Assuming the hypotheses (3) still valid, let us first consider the point spectrum associated with the problem, and look for solutions of the form:

$$P(\mathbf{x}, t) = Me^{\omega_n t}\phi_n(\mathbf{x}); \qquad \Theta(\mathbf{x}, t) = Ne^{\omega_n t}\phi_n(\mathbf{x}), \tag{28}$$

where we mean by ϕ_n the eigenfunctions of the problem, that is,

$$\begin{cases} \Delta\phi_n = -\lambda_n\phi_n & \text{in } \mathcal{B} \\ \phi_n = 0 & \text{on } \partial\mathcal{B} \end{cases} \tag{29}$$

and where all eigenvalues λ_n must be taken positive in view of the Dirichlet boundary conditions. Due to equations (28) and (29), ω_n must satisfy the following equations

$$\begin{cases} M\left(\omega_n^2+\omega_n+\lambda_n\right)-N\left(\omega_n^2+\omega_n-A\lambda_n\right)=0 \\ -M(\gamma-1)\left(\beta\omega_n^2+\omega_n\right)+N\left(\beta\gamma\omega_n^2+\gamma\omega_n+m^2\lambda_n\right)=0 \end{cases} \tag{30}$$

which, understood as a linear system in the unknowns M and N, will admit non-trivial solutions if

$$\left(\omega_n^2+\omega_n+\lambda_n\right)\left(\beta\gamma\omega_n^2+\gamma\omega_n+m^2\lambda_n\right)-(\gamma-1)\left(\beta\omega_n^2+\omega_n\right)\left(\omega_n^2+\omega_n-A\lambda_n\right)=0 \tag{31}$$

Equation (31) can be rewritten as

$$a_4\omega_n^4+a_3\omega_n^3+a_2\omega_n^2+a_1\omega_n+a_0=0, \tag{32}$$

where

$$\begin{aligned} a_0&=m^2\lambda_n^2 \\ a_1&=\left[m^2+\gamma+A(\gamma-1)\right]\lambda_n \\ a_2&=1+\left[m^2+\beta\gamma+A\beta(\gamma-1)\right]\lambda_n \\ a_3&=1+\beta \\ a_4&=\beta \end{aligned} \tag{33}$$

If we look for conditions capable of guaranteeing that the real parts of the solutions are negative, we can use the Routh–Hurwitz criterion, that is, we can verify that $a_i>0$, for $i=0,\dots,4$ (trivially satisfied in view of (3)) and that – following one of the possible check criteria – for a polynomial of degree four it results:

$$a_1a_2-a_0a_3>0 \tag{34}$$

together with

$$a_1a_2a_3-a_1^2a_4-a_0a_3^2>0 \tag{35}$$

In our case, actually, it is easy to prove that:

$$a_1a_2-a_0a_3=\Lambda_{21}\lambda_n+\Lambda_{22}\lambda_n^2>0 \tag{36}$$

and

$$a_1a_2a_3-a_1^2a_4-a_0a_3^2=\Lambda_{31}\lambda_n+\Lambda_{32}\lambda_n^2>0 \tag{37}$$

where

$$\begin{aligned} \Lambda_{21}&=m^2+\gamma+A(\gamma-1) \\ \Lambda_{22}&=m^4+m^2(1+\beta)(\gamma-1)(1+A)+\beta[\gamma+A(\gamma-1)]^2 \\ \Lambda_{31}&=(1+\beta)\left[m^2+\gamma+A(\gamma-1)\right] \\ \Lambda_{32}&=\left(m^2-\beta\right)^2+\beta^2(\gamma-1)(1+A)[\gamma+1+A(\gamma-1)] \\ &\quad+m^2\left(1+\beta^2\right)(\gamma-1)(1+A) \end{aligned} \tag{38}$$

and where we emphasize that the proposed expressions for (36), (37) (and (38)) have been rearranged in such a way as to highlight that, *under assumptions* (3) *alone*, both relations (>0) always remain satisfied[3].

[3] We remark that expressions given in (36) and (37) remain positive also when $A>-1$, but $m^2+A\beta\gamma>0$.

In order to deepen the spectral analysis, we will explore the possibility that Equation (32) admits solutions which are close (or not) to the complex axis. In other words we will try to understand if, given ϵ positive but small as desired, solutions exist to (32) which are on the left of the vertical line $\Re\{z\} = -\epsilon$. For this purpose, let us study the following polynomial equation:

$$a_4(\omega_n - \epsilon)^4 + a_3(\omega_n - \epsilon)^3 + a_2(\omega_n - \epsilon)^2 + a_1(\omega_n - \epsilon) + a_0 = 0, \tag{39}$$

which we can rewrite as

$$b_4\omega_n^4 + b_3\omega_n^3 + b_2\omega_n^2 + b_1\omega_n + b_0 = 0, \tag{40}$$

where

$$\begin{aligned} b_0 &= a_0 + P_0(\epsilon) \\ b_1 &= a_1 + P_1(\epsilon) \\ b_2 &= a_2 + P_2(\epsilon) \\ b_3 &= a_3 + P_3(\epsilon) \\ b_4 &= a_4 \end{aligned} \tag{41}$$

and where

$$\begin{aligned} P_0(\epsilon) &= a_4\epsilon^4 - a_3\epsilon^3 + a_2\epsilon^2 - a_1\epsilon \\ P_1(\epsilon) &= -4a_4\epsilon^3 + 3a_3\epsilon^2 - 2a_2\epsilon \\ P_2(\epsilon) &= 6a_4\epsilon^2 - 3a_3\epsilon \\ P_3(\epsilon) &= -4a_4\epsilon \end{aligned} \tag{42}$$

and resort to the Routh–Hurwitz criterion again. In this case, we have:

$$b_1b_2 - b_0b_3 = \epsilon M_{20} + (\Lambda_{21} - \epsilon M_{21})\lambda_n + (\Lambda_{22} - \epsilon M_{22})\lambda_n^2 \tag{43}$$

and

$$b_1b_2b_3 - b_1^2b_4 - b_0b_3^2 = \epsilon M_{30} + (\Lambda_{31} - \epsilon M_{31})\lambda_n + (\Lambda_{32} - \epsilon M_{32})\lambda_n^2 \tag{44}$$

where M_{20}, M_{21}, M_{22}, and M_{30}, M_{31}, M_{32} are polynomials depending on both the parameters and ϵ. By continuity, we can select ϵ_1 such that:

$$\begin{aligned} \Lambda_{21} - \epsilon M_{21} &\geq (1/2)\Lambda_{21} \\ \Lambda_{22} - \epsilon M_{22} &\geq (1/2)\Lambda_{22} \\ \Lambda_{31} - \epsilon M_{31} &\geq (1/2)\Lambda_{31} \\ \Lambda_{32} - \epsilon M_{32} &\geq (1/2)\Lambda_{32} \end{aligned} \tag{45}$$

for every $0 < \epsilon \leq \epsilon_1$. Also, we can select ϵ_2 small enough to satisfy $\epsilon M_{20} + (1/2)\,\Lambda_{21}\lambda_1 > 0$ and $\epsilon M_{30} + (1/2)\,\Lambda_{31}\lambda_1 > 0$ for every $\epsilon \leq \epsilon_2$.

In conclusion, we can select ϵ small enough to guarantee that both quantities given in (43) and (44) are positive. That is, we can guarantee that – for every domain $\mathcal{B}$ – there exists ϵ small enough, but positive, such that the point spectrum of our problem is on the left of the vertical line $\Re\{z\} = -\epsilon$. The fact that the point spectrum satisfies this property only suggests that the solutions to our problem are exponentially stable; however, the one proposed cannot be framed as a rigorous mathematical proof. So, our intention is to demonstrate the exponential decay of the solutions when conditions (3) and (14) hold, that is, when – we recall – also the function $D(t)$ given in (13) is positive definite. That is, there exists a positive

constant $\mathcal{K}$ such that

$$D(t) \geq \mathcal{K} \int_{\mathcal{B}} \left(P_t^2 + \Theta_t^2\right) dv \tag{46}$$

Now, let us multiply Equation $(4)_1$ by P and Equation $(4)_2$ by Θ. After integration we see

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{B}} \left\{ \left[1 + \frac{A}{m^2}\beta(\gamma-1)\right] P_t P + \frac{1}{2}\left[1 + \frac{A}{m^2}(\gamma-1)\right] P^2 - \left(1 + \frac{A}{m^2}\beta\gamma\right)\Theta_t P \right\} dv \\ &\quad = -\int_{\mathcal{B}} |\nabla P|^2 dv + \left[1 + \frac{A}{m^2}\beta(\gamma-1)\right]\int_{\mathcal{B}} P_t^2 dv \\ &\quad\quad -\left(1 + \frac{A}{m^2}\beta\gamma\right)\int_{\mathcal{B}} \Theta_t P_t dv + \left(1 + \frac{A}{m^2}\gamma\right)\int_{\mathcal{B}} \Theta_t P dv \\ &\quad \leq -\frac{1}{2}\int_{\mathcal{B}} |\nabla P|^2 dv + C_1 \int_{\mathcal{B}} \left(P_t^2 + \Theta_t^2\right) dv \end{aligned} \tag{47}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_{\mathcal{B}} \left[\beta\gamma\Theta_t\Theta + \frac{\gamma}{2}\Theta^2 - (\gamma-1)\beta P_t\Theta\right] dv \\ &\quad = -m^2\int_{\mathcal{B}} |\nabla\Theta|^2 dv + \beta\gamma\int_{\mathcal{B}} \Theta_t^2 dv - (\gamma-1)\beta\int_{\mathcal{B}} P_t\Theta_t dv + (\gamma-1)\int_{\mathcal{B}} P_t\Theta dv \\ &\quad \leq -\frac{m^2}{2}\int_{\mathcal{B}} |\nabla\Theta|^2 dv + C_2\int_{\mathcal{B}} \left(P_t^2 + \Theta_t^2\right) dv \end{aligned} \tag{48}$$

where C_1 and C_2 are two positive computable constants.

Now, if we define

$$G_1(t) = \int_{\mathcal{B}} \left\{ \left[1 + \frac{A}{m^2}\beta(\gamma-1)\right] P_t P + \frac{1}{2}\left[1 + \frac{A}{m^2}(\gamma-1)\right] P^2 - \left(1 + \frac{A}{m^2}\beta\gamma\right)\Theta_t P \right\} dv \tag{49}$$

and

$$G_2(t) = \int_{\mathcal{B}} \left[\beta\gamma\Theta_t\Theta + \frac{\gamma}{2}\Theta^2 - (\gamma-1)\beta P_t\Theta\right] dv \tag{50}$$

we can consider

$$\mathcal{E}(t) = E(t) + \epsilon\left[G_1(t) + G_2(t)\right] \tag{51}$$

and note that if ϵ is small enough, but positive, then $\mathcal{E}(t)$ is equivalent to $E(t)$, that is, there exist two positive constants $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 E(t) \leq \mathcal{E}(t) \leq \mathcal{M}_2 E(t)$.

On the other side, as we can select ϵ sufficiently small, we can see that

$$\frac{d}{dt}\mathcal{E}(t) \leq -k\int_{\mathcal{B}} \left(|\nabla P|^2 + |\nabla\Theta|^2 + P_t^2 + \Theta_t^2\right) dv \leq -k^* E(t) \leq -k^*\mathcal{M}_2^{-1}\mathcal{E}(t), \tag{52}$$

where again k and k^* are two calculable positive constants. It then follows

$$\frac{d}{dt}\left[e^{\omega t}\mathcal{E}(t)\right] \leq 0 \quad \Rightarrow \quad \mathcal{E}(t) \leq \mathcal{E}(0)e^{-\omega t}, \tag{53}$$

where $\omega = k^* \mathcal{M}_2^{-1}$. Therefore,

$$E(t) \leq \mathcal{M}_1^{-1}\mathcal{E}(t) \leq \mathcal{M}_1^{-1}\mathcal{E}(0)e^{-\omega t} \leq \mathcal{M}_2\mathcal{M}_1^{-1}E(0)e^{-\omega t} \tag{54}$$

Thus, we can conclude that there exist two further positive constants, $\mathcal{M} = \mathcal{M}_2\mathcal{M}_1^{-1}$ and ω, such that

$$E(t) \leq \mathcal{M}E(0)e^{-\omega t} \tag{55}$$

and so the exponential decay remains proven, whenever (3) and (14) hold.

Theorem 2. *Let the assumptions (3) and (14) be valid, and let* (P, Θ) *be a solution of the problem described by the system (2), with homogeneous boundary conditions (9). Then, such solution* (P, Θ) *decays exponentially over time.*

Remark 2. In light of the previous arguments, it is natural to expect the exponential decay under the sole condition (3), but it seems that we should intercept alternative energy functions by considering alternative weights for the equations of system (4). However, we leave this study to the readers.

5 | FINAL COMMENTS

The object of our investigation is represented by the linearized ECCS model for thermoporoacoustic phenomena given in ref. [1], but adapted to a three-dimensional domain identifying a rigid, stationary porous matrix permeated by a perfect gas. In ref. [1], the singular-surface theory has been applied to such model, which is why we felt the need to deep into its theoretical foundations, in terms of well-posedness and qualitative aspects of its solutions.

After giving a detailed description of the mathematical model and defining the related initial-boundary value problem (Section 2), we have resorted to the semigroup theory for linear operators (Section 3) in order to prove that – for the above problem – the solution exists, is unique, and depends continuously on the initial data and supply terms, in case they are considered (see Theorem 1 and subsequent Remark 1); in other words, we have demonstrated that the problem is well-posed in Hadamard's interpretation when the following assumptions (see (3)) are fulfilled: $\gamma > 1$, $A\,(= \sigma_0\vartheta_0/p_0) > 0$ and $\beta\,(= \tau_0\delta_D) > 0$. In the next Section (4), we shed light on the fact that the above solutions decay, also identify the quality of decay (exponential over time) under slightly more restrictive assumptions on the constitutive coefficients (see (3) and (14)) – refer in this regard to Theorem 2 and subsequent Remark 2. The Routh–Hurwitz criterion was used for this purpose.

It is worth highlighting that the element of novelty introduced with respect to the existing literature is represented by the mathematical model investigated and described by the system of partial differential equations (2) (or (4) or even (8)): with regard to it, well-established techniques are employed to study – on one side – the existence, uniqueness, and continuous dependence of the solution, and – on the other – the decay of solutions. Among the elements worthy of attention, even in future works, there is certainly the identification of the weakest restrictions capable of ensuring that such model leads to a well-posed problem and that the solutions decay exponentially over time.

We believe that this contribution can be important to consolidate a model for which interesting results have already been obtained performing a singular-surface analysis, and that is proving to be very promising with reference to the possibility of modelling numerous engineering questions (see Section 1).

ACKNOWLEDGMENTS

V.Z. was supported by funding provided by the Italian National Group of Mathematical Physics (GNFM–INdAM), by the NextGenerationEU PRIN2022 research project *The Mathematics and Mechanics of Nonlinear Wave Propagation in Solids* (grant no. 2022P5R22A; CUP D53D23003890006), and by the National Recovery and Resilience Plan (NRRP) NextGenerationEU research project *SUSTBUILD–SUSTainable composite structures for energy-harvesting and carbon-storing BUILDings* (grant no. P2022PE8BT; CUP D53D23018440001).

Open access publishing facilitated by Universita degli Studi di Salerno, as part of the Wiley - CRUI-CARE agreement.

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How to cite this article: Quintanilla, R., Zampoli, V.: On the linearized Eringen–Cattaneo–Christov–Straughan model: Some qualitative properties of solutions. Z Angew Math Mech. e202400869 (2024). https://doi.org/10.1002/zamm.202400869